Tilastollisen päättelyn jatkokurssi, sl 2010, Exercise 6, week 42

1. Show that the hypothesis $A\theta_0 = c$, where the matrix A $(q \times d)$ and the vector c $(q \times 1)$ are known and $\mathsf{r}(A) = q$, is equivalent to the hypothesis $\theta_0 = B\delta_0 + e$, where the matrix B $(d \times (d-q))$ and the vector e $(d \times 1)$ are known and $\mathsf{r}(B) = d - q$.

Hint: As the rows of the matrix A are linearly independent, we know from linear algebra that there exist linearly independent vectors $m_1, ..., m_{d-q}$ $(d \times 1)$ such that the $d \times d$ matrix $P = [A' : M]$, where $M = [m_1 : \cdots : m_{k-l}]$, is nonsingular and the column vectors of the matrices A' and M are orthogonal so that $AM = 0$.

2. Continuation to exercises 4.1 and 5.2. Consider the hypothesis $\mathsf{H}_0 : \beta_0 = 0$ under which the explanatory variable x_i has no effect.

(i) Derive the score function of the parameter $\theta = (\alpha, \beta)$ and the asymptotic distribution of the unconstrained maximum likelihood estimator $\hat{\theta}$ when the consistency of $\hat{\theta}$ and needed convergence results are assumed to hold.

(ii) Derive the Wald test for the hypothesis H_0 against the alternative $\beta_0 \neq 0$.

(iii) Use the preceding Wald test to derive a 95% confidence interval for the parameter β.

Note: In this case it is not possible to obtain an analytic solution for the maximum likelihood estimator $\hat{\theta}$. The confidence interval in part (iii) can be obtained in the same way as in the first course on statistical inference (Tilastollisen päättelyn kurssi).

3. Continuation to the preceding one. Derive the constrained maximum likelihood estimator $\tilde{\theta}$ of the parameter $\theta = (\alpha, \beta)$ (that is, constrained by H_0).

(ii) What is the asymptotic distribution of the constrained maximum likelihood estimator $\tilde{\theta}$?

(iii) Derive Rao's score test for the hypothesis H_0 against the alternative $\beta_0 \neq 0$ when the convergence results assumed in the preceding exercise hold.

Note: The constrained maximum likelihood is considered on p. 38 of the lecture notes and also in the first course on statistical inference (Tilastollisen päättelyn kurssi).

4. Continuation to the two preceding ones. Consider the same testing problem but assuming now that the alternative hypothesis $\beta_0 > 0$. Derive the Wald test and Rao's score test in this case.

Note: One-sided alternatives are not considered in the lecture notes but at least in the Finnish course material used in the first course on statistical inference (Tilastollisen päättelyn kurssi) they are treated with sufficient detail.