Tilastollisen päättelyn jatkokurssi, sl 2010, Exercise 5, week 41

1. Continuation to exercises 3.4 and 4.2. (i) Derive the maximum likelihood (ML) estimate of the parameter vector $\theta = (\phi, \sigma^2)$, when the parameter space is $(\phi, \sigma^2) \in \mathbb{R} \times (0, \infty)$.
(ii) Justify why, $\hat{\phi}$, the ML estimator of ϕ is not unbiased, that is, $\mathsf{E}_\theta(\hat{\phi}) \neq \phi$ (exact mathematical proof is not required).

Note: Part (ii) has implications on the linear model defined in equation (2.16) (p. 21) of the lecture notes (a linear version of the nonlinear model in exercise 4.3). In particular, it demonstrates that the usual theory of ML estimation (and hence also the usual theory of F–test considered in the last exercise below) does not hold if the vector of explanatory variables contains previous values of Y_i (e.g., Y_{i-1}) or, more generally, if the condition $(\boldsymbol{W}_{i-1}, X_i) \perp\!\!\!\perp \varepsilon_i$ does not hold in equation (2.16) of the lecture notes.

2. (i) Show that the score function of exercise 4.1 is a martingale with respect to the information set $\mathbf{Y}_n = (Y_1, ..., Y_n)$.

(ii) Show that the score function of exercises 3.4 and 4.2 is a martingale with respect to the information set $\mathbf{Y}_n = (Y_1, ..., Y_n)$.

(ii) Show that the score function of exercise 4.3 is a martingale with respect to the information set $(\mathbf{Y}_n, \boldsymbol{Z}_{n+1}) = (Y_1, ..., Y_n, Z_1, ..., Z_{n+1})$ when $\mathsf{E}\left[\|\partial g\,(z;\beta)\,/\partial\beta\|\right] < \infty$.

Hint: See exercise 3.1 and note that the definition of a martingale is given by conditions (i), (ii) and (iii) on p. 10 of the lecture notes.

3. Consider the linear model defined in equation (2.16) (p. 21) of the lecture notes and let $\hat{\beta}$ and $\hat{\sigma}^2$ be the related ML estimators defined on p. 21-22 of the lecture notes. Assume the vector of explanatory variables, Z_i, satisfies condition (2.26) of the lecture notes, that is, $n^{-1}\sum_{i=1}^n Z_i Z_i' \xrightarrow{p} Q$, where Q is a positive definite (and hence nonsingular) matrix. Assume further that $\mathsf{E}(Z_{a,i}^2) \leq C < \infty$ for all $a = 1, ..., p$ and $i \geq 1$.

(i) Show that the ML estimator $\hat{\sigma}^2$ is consistent, that is, $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$.

(ii) Now consider the nonlinear extension of the model introduced in exercise 4.3 and let $\hat{\sigma}^2$ be the ML estimator of σ^2 in this model. Suppose it is known that $\hat{\beta}$, the ML estimator β, is consistent. Present a sufficient condition which implies the consistency of the ML estimator $\hat{\sigma}^2$ in this case.

Hint: In part (i) you can write $Y_i - Z_i'\hat{\beta} = (Y_i - Z_i'\beta) + Z_i'(\hat{\beta} - \beta)$ in the expression of $\hat{\sigma}^2$. Taking the square on both sides of this equation, summing over $i = 1, ..., n$ and dividing the sum by n you get an expansion for the estimator $\hat{\sigma}^2$ which can be used to establish the consistency. In the terms of the expansion you can use the law of large numbers, the mentioned condition (2.26), and the consistency of $\hat{\beta}$ established on p. 26-27 of the lecture notes (so you may assume the consistency of $\hat{\beta}$ here).

4. Consider the same linear model as in the preceding exercise and assume that the ML (or least squares) estimator $\hat{\beta}$ is asymptotically normal or that

$$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} \mathsf{N}\left(0, \sigma^2 Q^{-1}\right).$$

Assume the parameter β $(p \times 1)$ is constrained by the linear hypothesis $\mathsf{H}_0 : A\beta = c$ where the matrix A $(q \times p)$ and the vector c $(q \times 1)$ are known and the rank of A is q. The alternative hypothesis is $A\beta \neq c$. Below the hypothesis H_0 is assumed to hold.

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(i) Show that $\sqrt{n}(A\hat{\beta} - c) \xrightarrow{d} \mathsf{N}\left(0, \sigma^2 A Q^{-1} A'\right)$.

(ii) The F–statistic considered in the course of linear models can be expressed as

$$F = n(A\hat{\beta} - c)' \left[A \left(n^{-1} \sum\nolimits_{i=1}^{n} Z_i Z_i' \right)^{-1} A' \right]^{-1} (A\hat{\beta} - c)/qS^2,$$

where $S^2 = n\hat{\sigma}^2 / (n - p)$. Show that $qF \xrightarrow{d} \chi^2_q$.

Hint: In part (i), Theorem 1.3 and properties of the multinormal distribution (see the hint to exercise 3.5). In part (ii), part (i), condition (2.26) and the consistency of S^2 as an estimator of σ^2 (cf. exercise 2.3(ii)) combined with Theorem 1.1, Corollary 1.2, and the hint of exercise 1.5. (Here Theorems 1.1 and 1.3 (Lause 1.1 and 1.2) as well as Corollary 1.2 (Seuraus (1.2) refer to the lecture notes.)