Tilastollisen päättelyn jatkokurssi, sl 2010, Exercise 3, week 39

1. Let $\{X_i\}_{i=1}^{\infty}$ be a real valued iid–sequence with $\mathsf{E}(X_1) = 0$ and define $\mathcal{F}_i = (X_1, ..., X_i)$.

(i) Assume that $\{X_i\}_{i=1}^{\infty}$ and $\{Z_i\}_{i=1}^{\infty}$ (real-valued) are independent and $\mathsf{E}(|Z_i|) < \infty$ for all $i \geq 1$. Show that $\{Z_i X_i\}_{i=1}^{\infty}$ is a martingale difference (MD) sequence with respect to the information set $\mathcal{F}_i = (X_1, ..., X_i, Z_1, ..., Z_{i+1})$.

(ii) Let $Z_1 = c$ (constant) and $Z_i = g_i(X_1, ..., X_{i-1})$, $i \geq 2$, such that $\mathsf{E}(|Z_i|) < \infty$ for all $i \geq 2$. Show that $\{Z_i X_i\}_{i=1}^{\infty}$ a MD sequence with respect to the information set $\mathcal{F}_i = (X_1, ..., X_i)$.

Note: The definition of a MD sequence is given by conditions (i), (ii) and (iii) on p. 11 of the lecture notes.

2. Let $X_1, X_2, ...$ $(k \times 1)$ be a vector valued MD sequence whose components have finite second moments and $M_n = X_1 + \cdots + X_n$. Assume further that $\mathsf{E}(X_1) = 0$. Show in detail that $\mathsf{Cov}(M_n) = \sum_{i=1}^{n} \mathsf{Cov}(X_i)$ or, in words, that a MD sequence is uncorrelated.

Note: The sequence $M_1, M_2, ...$ is a martingale. A general definition of a martingale is given by conditions (i), (ii) and (iii) on p. 10 of the lecture notes.

3. Assume the random vectors X and Z $(k \times 1)$ are independent and $Z \sim \mathsf{N}_k(\mu, \Sigma)$. Show that the conditional distribution of the random vector $Y = X + Z$ given $X = x$ is $\mathsf{N}_k(\mu + x, \Sigma)$ or, in symbols, $Y|(X = x) \sim \mathsf{N}_k(\mu + x, \Sigma)$.

Hint: You may assume that the distribution of X in continuous (the result holds without this assumption). Consider the linear transformation $(Z, X) \mapsto (Y, X)$ and derive the joint density function of (Y, X) using a well-known formula on the distribution of transformed random vectors (see, e.g., Koistinen: *Todennäköisyyslaskennan kurssin luentomoniste*, 11. joulukuuta 2009, p. 119–120)[1]. After this, you get the conditional density function in question by using the definition of the conditional density function and the independence $X \perp\!\!\!\perp Z$.

4. Consider the autoregressive time series model defined in equation (2.1) of the lecture notes (p. 13) and justify the expression of the related joint density function of $\mathbf{Y} = (Y_1, ..., Y_n)$ given on page 17 of the lecture notes, that is,

$$f_{\mathbf{Y}}(\boldsymbol{y}; \theta) = \left(2\pi\sigma^2\right)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \phi y_{i-1})^2\right\}, \quad \theta = \left(\phi, \sigma^2\right).$$

Hint: First derive the joint density $f_{Y_i, \mathbf{Y}_{i-1}}$ of $(Y_i, \mathbf{Y}_{i-1})$ by using the (one-to-one) transformation $(\varepsilon_i, \mathbf{Y}_{i-1}) \mapsto (Y_i, \mathbf{Y}_{i-1})$ and the related formula mentioned in the hint of exercise 2 (this transformation is linear and the Jacobian is unity). After this (if not earlier) use the independence of $\mathbf{Y}_{i-1} \perp\!\!\!\perp \varepsilon_i$ and the formula $f_{Y_i|\mathbf{Y}_{i-1}} = f_{Y_i, \mathbf{Y}_{i-1}} / f_{\mathbf{Y}_{i-1}}$.

5. Consider the following generalization of the model (2.2) (see the lecture notes)

$$Y_j = Z_j \beta_j + \varepsilon_j, \qquad j = 1, ..., N,$$

where Z_i $(n_j \times p)$ is a fixed (nonrandom) matrix of explanatory variables and $\varepsilon_1, ..., \varepsilon_N$ $\perp\!\!\!\perp$, $\varepsilon_j \sim \mathsf{N}_{n_j}\left(0, \sigma^2 I_{n_j}\right)$. Assume the parameter vector β_j $(p \times 1)$ is random with the properties

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[1] http://wiki.helsinki.fi/pages/viewpage.action?pageId=48308040

$$\beta_1, ..., \beta_N \perp\!\!\!\perp, \ \beta_j \sim \mathsf{N}_p(\beta, \Omega) \quad \text{and} \quad (\beta_1, ..., \beta_N) \perp\!\!\!\perp (\varepsilon_1, ..., \varepsilon_N).$$

Derive the joint density function of $\mathbf{Y}_n = (Y_1, ..., Y_n)$ by modifying the considerations given for model (2.2) (p. 14-15 of the lecture notes).

Hint: Here some basic properties of the multinormal distribution are assumed (especially that the sum of two multinormal independent random vectors with the same dimension is multinormal and also linear transformations of multinormally distributed random vectors (see, e.g., Koistinen: *Todennäköisyyslaskennan kurssin luentomoniste*, 11. joulukuuta 2009, Chapter 9).