Tilastollisen päättelyn jatkokurssi, sl 2010, Exercise 1, week 37

1. (i) Assume $0 \leq Y_n \leq Z_n$ and $Z_n \xrightarrow{p} 0$. Show that $Y_n \xrightarrow{p} 0$. Here Y_n and Z_n are real-valued.
(ii) Assume now that Y_n and Z_n are $k \times 1$ random vectors and $0 \leq \|Y_n\| \leq \|Z_n\|$ and $Z_n \xrightarrow{p} 0$. Show that $Y_n \xrightarrow{p} 0$.

Hint: Use the definition of stochastic convergence which the notation $\xrightarrow{p}$ signifies.

2. Let $\Theta \subseteq \mathbb{R}$ be an open interval, $h : \Theta \to \mathbb{R}$ continuously differentiable and $h'(\theta_0) \neq 0$. Let $\hat{\theta}_n$ be an asymptotically normally distributed estimator of the parameter θ_0 that satisfies

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{d} \mathsf{N}\left(0, \sigma^2(\theta_0)\right), \quad \sigma^2(\theta_0) > 0. \qquad (*)$$

Show that the estimator $\hat{\theta}_n$ is consistent, that is, $\hat{\theta}_n \xrightarrow{p} \theta_0$.

Hint: Apply Theorems (= Lause) 1.4 and 1.3 or Corollary (= Seuraus) 1.1 and after that the result (assumed known) that the convergence in distribution $U_n \xrightarrow{d} c$ (c nonrandom) implies the stochastic convergence $U_n \xrightarrow{p} c$. For possible later purposes note that θ_0 signifies the true value of the parameter θ.

3. (Continuation for the preceding one; the so-called delta method) Show that

$$\sqrt{n}(h(\hat{\theta}_n) - h(\theta_0)) \xrightarrow{d} \mathsf{N}\left(0, [h'(\theta_0)]^2\sigma^2(\theta_0)\right).$$

Hint: First conclude from the mean value theorem that $h(\hat{\theta}_n) - h(\theta_0) = h'(\bar{\theta}_n)(\hat{\theta}_n - \theta_0)$ with $\bar{\theta}_n = c\hat{\theta}_n + (1-c)\theta_0$, $0 \leq c \leq 1$. Use then the result $\hat{\theta}_n \xrightarrow{p} \theta_0$ from the preceding exercise to obtain $\bar{\theta}_n \xrightarrow{p} \theta_0$ and apply thereafter the asymptotic normality $\hat{\theta}_n$ (see $(*)$) in conjunction with Theorems 1.1, 1.4 and 1.3 or Corollary 1.1.

4. (Continuation for the two preceding ones). Derive the asymptotic distribution of

$$\frac{\sqrt{n}(h(\hat{\theta}_n) - h(\theta_0))}{h'(\hat{\theta}_n)\sigma(\hat{\theta}_n)}$$

when the function $\theta \longmapsto \sigma^2(\theta)$ is continuous at θ_0.

Hint: Apply the same theorems as the preceding exercises.

5. Assume

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{d} \mathsf{N}_d\left(0, \Sigma(\theta_0)\right),$$

where $\Sigma(\theta_0)$ is positive definite (and hence nonsingular having an inverse). Assume further that the function $\theta \longmapsto \Sigma(\theta)$ is continuous at θ_0. Show that

$$W = n(\hat{\theta}_n - \theta_0)'\Sigma(\hat{\theta}_n)^{-1}(\hat{\theta}_n - \theta_0) \xrightarrow{d} \chi^2_d.$$

Hint: In addition to the theorems that are used in the preceding exercises you can here also use the result $Z'\Sigma^{-1}Z \sim \chi^2_k$, when $Z \sim \mathsf{N}_k(0, \Sigma)$ and Σ $(k \times k)$ on positive definite (this result is assumed known from the course of linear models).

References for p. 1-15 of the lecture notes:

C.R. Rao "Linear Statistical Inference and Its Applications. Second Edition" (Wiley, 1973) Chapter 2.c

R.J. Serflingin "Approximation Theorems of Mathematical Statistics" (Wiley, 1980) Chapter 1 (or p. 1-30).