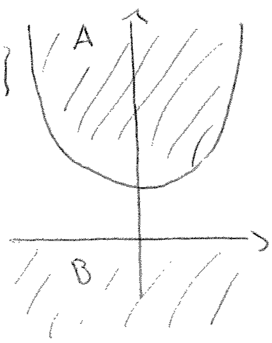
Topologia I, harj. 3

1. $A = \{(x,y) : x^2+1-y<0\}$, $B = \{(x,y) : y<0\}$



(a) $(x,y) \in A \Rightarrow y > x^2+1 \geq 1$

siis $\forall a=(a_1,a_2) \in A$, $b=(b_1,b_2)\in B$

pätee $d(a,b) = \sqrt{\underbrace{(a_1-b_1)^2}_{\geq 0} + \underbrace{(a_2-b_2)^2}_{\geq 1}} \geq 1$

siten $d(A,B) \geq 1$

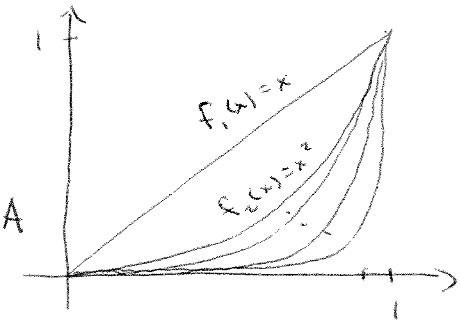
toisaalta $d(\underbrace{(0,1+t)}_{\in A}, \underbrace{(0,-t)}_{\in B}) = 1+2t \quad \forall t>0$

joten $d(A,B) \leq 1$

siispä $d(A,B)=1$

(b) $A \cap B = \emptyset \Rightarrow d(a,b)=1 \quad \forall a\in A, b\in B \Rightarrow d(A,B)=1$

2. $f_n(x) = x^n$, $x\in[0,1]$, $A=\{f_n : n\in\mathbb{N}\}$



$f\in A \Rightarrow 0\leq f(x)\leq 1 \quad \forall x\in[0,1]$

siis $\|f-g\| = \sup_{x\in[0,1]} \underbrace{|f(x)-g(x)|}_{\leq 1} \leq 1 \quad \forall f,g\in A$

eli $d(A)\leq 1$

toisaalta

olk. $0<t<1 \Rightarrow f_n(1-t/2) = (1-t/2)^n \to 0$, kun $n\to\infty$, sillä $|1-t/2|<1$

$\Rightarrow \exists k\in\mathbb{N} : |f_k(1-t/2)| < t/2$

$\Rightarrow \|f_1 - f_k\| \geq |1-t/2 - f_k(1-t/2)| \geq 1-t/2 - t/2 = 1-t$

siis $\forall t\in(0,1)\ \exists f,g\in A : \|f-g\| \geq 1-t$

$\Rightarrow d(A)\geq 1-t \quad \forall t\in(0,1) \Rightarrow d(A)\geq 1$

täten $d(A)=1$

3. X metrinen av., $G \subset X$ avoin, $F \subset G$ äärellinen $\Rightarrow G \setminus F$ avoin

<u>tod.</u>

olk. $x \in G \setminus F$

G avoin $\Rightarrow \exists r > 0 : B(x,r) \subset G$

F äärellinen $\Rightarrow d(x,F) = \min\limits_{y \in F} d(x,y) > 0$

val. $s = \min\{r, d(x,F)\} > 0 \Rightarrow B(x,s) \subset G \setminus F$ $\square$

4. merk. erakkopisteiden joukkoa A:lla

$\{a\}$ avoin $\forall a \in A \Rightarrow A = \bigcup\limits_{a \in A} \{a\}$ avoin

5. äärellinen metrinen avaruus on diskreetti

<u>tod.</u>

olk. (X,d) äärellinen metrinen av.

merk. $\forall x \in X$, $r_x = \min\{d(x,y) : y \in X \setminus \{x\}\} > 0$

$\Rightarrow B(x, r_x) = \{x\}$ $\forall x \in X$, eli $\{x\}$ on avoin $\forall x \in X$

jos siis $A \subset X$, niin $A = \bigcup\limits_{x \in A} \{x\}$ on avoin, ts. X on diskreetti $\square$

6. $x \in X \overset{Y \text{ diskreetti}}{\Rightarrow} \{f(x)\}$ avoin $\Rightarrow \exists \varepsilon > 0 : B(f(x), \varepsilon) = \{f(x)\}$

$\overset{f \text{ jatk.}}{\Rightarrow} \exists \delta > 0 : fB(x,\delta) \subset B(f(x), \varepsilon) = \{f(x)\}$

$\Rightarrow$ ~~f(y)~~ $f(x') = f(x)$ $\forall x' \in B(x, \delta)$