Topologia I, harj. 2

1. $\|x\|_1 = \sum_{k=1}^{n} |x_k|$, $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, normi $\mathbb{R}^n$:ssä

(N1) $\|x+y\|_1 = \sum_{k=1}^{n} |x_k+y_k| \le \sum_{k=1}^{n} (|x_k|+|y_k|) = \sum_{k=1}^{n}|x_k| + \sum_{k=1}^{n}|y_k|$

$= \|x\|_1 + \|y\|_1$

(N2) $\|ax\|_1 = \sum_{k=1}^{n} |ax_k| = \sum_{k=1}^{n} |a||x_k| = |a| \sum_{k=1}^{n}|x_k| = |a|\,\|x\|_1$

(N3) $\|x\|_1 = 0 \iff \sum_{k=1}^{n} |x_k| = 0 \underset{\substack{|x_k|\ge 0\\ k=1,\dots,n}}{\iff} |x_k| = 0,\ k=1,\dots,n$

$\iff x_k = 0,\ k=1,\dots,n \iff x=0$

2. $E$ sisätuloav., $x,y\in E$

$\Rightarrow \|x+y\|^2 + \|x-y\|^2 = (x+y)\cdot(x+y) + (x-y)\cdot(x-y)$

$= x\cdot x + x\cdot y + y\cdot x + y\cdot y + x\cdot x - x\cdot y - y\cdot x + y\cdot y$

$= 2\|x\|^2 + 2\|y\|^2$

$\|e_1+e_2\|_1^2 + \|e_1-e_2\|_1^2 = (1+1)^2 + (1+1)^2 = 8$

$2\|e_1\|_1^2 + 2\|e_2\|_1^2 = 2\cdot 1^2 + 2\cdot 1^2 = 4 \neq 8$

3. $\|f\|_1 = \int_0^1 |f(x)|\,dx$ normi $C[0,1]$:ssä [hyvin määritelty, sillä $f$ Riemann-integroituva]

(N1) $\|f+g\|_1 = \int_0^1 |f(x)+g(x)|\,dx \le \int_0^1 (|f(x)|+|g(x)|)\,dx$

$= \int_0^1 |f(x)|\,dx + \int_0^1 |g(x)|\,dx = \|f\|_1 + \|g\|_1$

(N2) $\|af\|_1 = \int_0^1 |af(x)|\,dx = |a| \int_0^1 |f(x)|\,dx = |a|\,\|f\|_1$

(N3) $f=0 \Rightarrow \int_0^1 \underbrace{|f(x)|}_{=0}\,dx = 0$

$f\neq 0 \Rightarrow \exists x_0\in[0,1] : |f(x_0)|>0$

$\overset{f \text{ jatk.}}{\Rightarrow} \exists \delta>0 : |f(x)-f(x_0)| < \frac{|f(x_0)|}{2}\ \forall x\in[0,1],\ |x-x_0|<\delta$

$\Rightarrow \exists [a,b]\subset[0,1] : |f(x)| > \frac{|f(x_0)|}{2}\ \forall x\in[a,b],\ a<b$

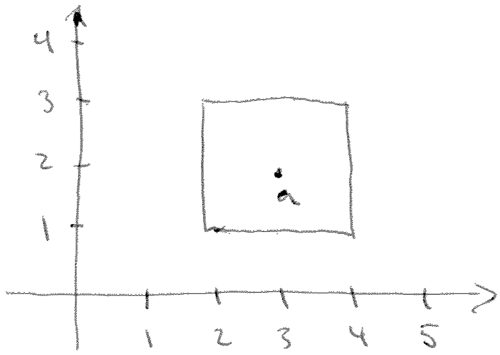
$\Rightarrow \int_0^1 |f(x)|\,dx \ge \int_a^b |f(x)|\,dx \ge (b-a)\frac{|f(x_0)|}{2} > 0$

4. $S(a,1)$ normissa $\|x\|_\infty = \max\{|x_1|, |x_2|\}$, $x=(x_1,x_2)\in\mathbb{R}^2$, kun $a=(3,$

$\|x-a\|_\infty = 1 \iff \max\{|x_1-3|, |x_2-2|\} = 1$

$\iff \big(|x_1-3| = 1$ ja $|x_2-2| \le 1\big)$ tai $\big(|x_1-3| \le 1$ ja $|x_2-2| = 1\big)$

$\iff \big((x_1=4$ tai $x_1=2)$ ja $1\le x_2\le 3\big)$ tai $\big(2\le x_1\le 4$ ja $(x_2=3$ tai $x_2=1)\big)$




5. $E$ normiav., os. $d(x,y) = \begin{cases} \|x\|+\|y\|, & x\ne y \\ 0, & x=y \end{cases}$ metriikka $E$:ssä

olk. $x,y,z\in E$:

(M1) jos $x=y$ tai $y=z$ tai $x=z$, niin asia selvä

ol. $x\ne y$, $y\ne z$, $x\ne z$

$\Rightarrow d(x,z) = \|x\|+\|z\| \le \|x\|+\|y\|+\|y\|+\|z\| = d(x,y)+d(y,z)$

(M2) jos $x=y$, niin asia selvä

ol. $x\ne y \Rightarrow d(x,y) = \|x\|+\|y\| = \|y\|+\|x\| = d(y,x)$

(M3) $d(x,x)=0$ määritelmästä

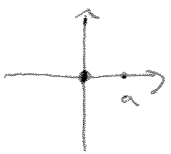
$x\ne y \Rightarrow x\ne 0$ tai $y\ne 0 \Rightarrow d(x,y) = \|x\|+\|y\| > 0$

$S(a,r)$, kun $E=\mathbb{R}^2$, $a=2e_1$ ja $r=1,2,3,4,5$

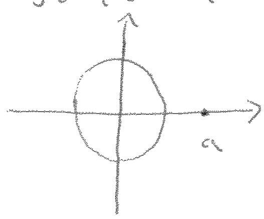
$d(x,a) = r \iff \|x\|+2 = r$

$S(a,1) = \emptyset$

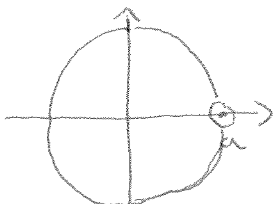
$S(a,2) = \{0\}$



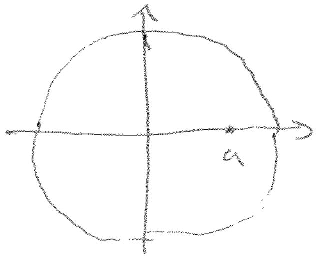

$S(a,3) = \{(x_1,x_2)\in\mathbb{R}^2 : x_1^2+x_2^2 = 1\}$




$S(a,4) = \{(x_1,x_2)\in\mathbb{R}^2 : x_1^2+x_2^2=4\}\setminus\{a\}$




$S(a,5) = \{(x_1,x_2)\in\mathbb{R}^2 : x_1^2+x_2^2 = 9\}$



6. d ja e metriikoja X:ssä

os. $d+e$ metriikka:

olk. $x,y,z \in X$

(M1) $(d+e)(x,z) = d(x,z) + e(x,z) \leq d(x,y) + d(y,z) + e(x,y) + e(y,z)$

$= d(x,y) + e(x,y) + d(y,z) + e(y,z)$

$= (d+e)(x,y) + (d+e)(y,z)$

(M2) $(d+e)(x,y) = d(x,y) + e(x,y) = d(y,x) + e(y,x) = (d+e)(y,x)$

(M3) $(d+e)(x,y) = 0 \iff d(x,y) + e(x,y) = 0 \underset{\substack{d(x,y) \text{ ja} \\ e(x,y) \geq 0}}{\iff}$ $d(x,y)=0$ ja $e(x,y)=$

$\iff x = y$

os. $d \vee e$ metriikka:

olk. $x,y,z \in X$

(M1) $(d \vee e)(x,z) = \max\{d(x,z), e(x,z)\}$

$\leq \max\{d(x,y) + d(y,z), e(x,y) + e(y,z)\}$

$\leq \max\{d(x,y), e(x,y)\} + \max\{d(y,z), e(y,z)\}$

$= (d \vee e)(x,y) + (d \vee e)(y,z)$

(M2) $(d \vee e)(x,y) = \max\{d(x,y), e(x,y)\} = \max\{d(y,x), e(y,x)\}$

$= (d \vee e)(y,x)$

(M3) $(d \vee e)(x,y) = 0 \iff \max\{d(x,y), e(x,y)\} = 0$

$\iff d(x,y) = 0,\ e(x,y) = 0 \iff x = y$