Introduction to Mathematical Physics: Spectral Theory Homework set 9

26. Nov 2010

Exercise 1

Consider a topological vector space $\mathcal{V}$ and some $E \subset \mathcal{V}$. Prove the following statements.

(a) If E is totally bounded, it is (topologically) bounded.

(b) If E is compact, it is totally bounded.

(c) Assume, in addition, that the topology of $\mathcal{V}$ is induced by an *invariant* metric d. Then E is totally bounded if and only if for any $r > 0$ one can cover E with a finite number of balls of radius r (*i.e.*, there should be $N \in \mathbb{N}_+$ and $x_i \in \mathcal{V}$, $i = 1, \ldots, N$, such that $E \subset \bigcup_{i=1}^N B(x_i, r)$).

Exercise 2

Let $\mathcal{B}$ be a Banach space and suppose $f : [0,1] \to \mathcal{B}$ is continuous. For $n \in \mathbb{N}_+$, define as vector valued integrals

$$\psi_n := \int_0^1 \mathrm{d}t \, \frac{n}{1 + n^2 t^2} f(t) \, .$$

Show that each ψ_n is well-defined, and that the sequence (ψ_n) converges in norm. What is the limit? (Hint: It might be instructive to look at the weak limit first.)

Exercise 3

Let μ be a Borel probability measure on a compact Hausdorff space Q, and suppose $\mathcal{F}$ is a Fréchet space and $f : Q \to \mathcal{F}$ is continuous. Show that the vector valued integral $\int_Q \mathrm{d}\mu f$ can then be "approximated by Riemann sums" in the following sense: for any neighborhood V_0 of zero there exists a partition of Q such that the difference between the integral and the Riemann sums belongs to V_0. Explicitly, you need to prove that if $V_0 \in \mathcal{T}$ with $0 \in V_0$, there are $N \in \mathbb{N}_+$ and Borel sets $E_i \subset Q$, $i = 1, \ldots, N$, such that $E_i \cap E_j = \emptyset$ for $i \neq j$, $\bigcup_{i=1}^N E_i = Q$, and

$$\sum_{i=1}^N \mu(E_i) f(E_i) \subset \int_Q \mathrm{d}\mu f + V_0 \, .$$

The last condition is equivalent to requiring $\sum_{i=1}^N \mu(E_i) f(t_i) - \int_Q \mathrm{d}\mu f \in V_0$ for any choice of $t_i \in E_i$, $i = 1, \ldots, N$. (Hint: You can assume V_0 is convex and balanced. Choose the sets E_i so that $f(s) - f(t) \in V_0$ whenever $s, t \in E_i$. If z denotes the above difference, then for any $\Lambda \in \mathcal{F}^*$ with $|\Lambda x| \leq 1$ for $x \in V_0$ one has $|\Lambda z| \leq 1$.)

(Please turn over!)

Exercise 4

Let $C := \{(\cos\varphi, \sin\varphi, 0) \in \mathbb{R}^3 \mid \varphi \in [0, 2\pi]\}$ be the unit circle in the xy-plane, and define $E := C \cup \{\mathbf{r}, \mathbf{r}'\} \subset \mathbb{R}^3$ with $\mathbf{r} = (1, 0, 1)$ and $\mathbf{r}' = (1, 0, -1)$. Consider $K = \mathrm{Hull}(E)$, and let S_0 denote the collection of extreme points of K. Show that K is compact but S_0 is not compact. Does such an example exist in $\mathbb{R}^2$?

Exercise 5

Consider the closed unit balls B_p in $L^p([0,1])$ (relative to the Lebesgue measure) for $1 \leq p < \infty$, i.e., define $B_p := \{f \in L^p \mid \|f\|_p \leq 1\}$.

(a) Show that $B_1 \subset L^1$ has no extreme points.

(b) Show that every $f \in L^p$ with $\|f\|_p = 1$ is an extreme point of B_p if $1 < p < \infty$.