Introduction to Mathematical Physics: Spectral Theory

Homework set 8
19. Nov 2010

Exercise 1

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be normed spaces. Define

$$\mathcal{B}(\mathcal{V}_1, \mathcal{V}_2) := \{T : \mathcal{V}_1 \to \mathcal{V}_2 \,|\, T \text{ is linear and continuous}\} .$$

(a) Show that a linear map $T : \mathcal{V}_1 \to \mathcal{V}_2$ belongs to $\mathcal{B}(\mathcal{V}_1, \mathcal{V}_2)$ if and only if $\|T\| < \infty$, where

$$\|T\| := \sup_{x \in \mathcal{V}_1,\, \|x\| \leq 1} \|Tx\| .$$

(b) Prove that $\mathcal{B}(\mathcal{V}_1, \mathcal{V}_2)$ is a vector space and $T \mapsto \|T\|$ defines a norm on it.

(c) Show that, if $\mathcal{V}_2$ is a Banach space, then $\mathcal{B}(\mathcal{V}_1, \mathcal{V}_2)$ is a Banach space.

Exercise 2

Consider a normed space $\mathcal{V}$ and its dual space $\mathcal{V}^*$. For every $\Lambda \in \mathcal{V}^*$ define

$$\|\Lambda\| := \sup_{x \in \mathcal{V},\, \|x\| \leq 1} |\Lambda x| .$$

Conclude that $\Lambda \mapsto \|\Lambda\|$ makes $\mathcal{V}^*$ into a Banach space. Set

$$B^* := \{\Lambda \in \mathcal{V}^* \,|\, \|\Lambda\| \leq 1\} ,$$

and recall the notation f_x for the map $f_x(\Lambda) = \Lambda(x)$ and that by Proposition 13.10 each f_x is a linear functional on $\mathcal{V}^*$.

(a) Show that $\|x\| = \sup_{\Lambda \in B^*} |\Lambda x|$, for any $x \in \mathcal{V}$.

(b) Show that every f_x is norm-continuous, and $\|f_x\| = \|x\|$.

(c) Show that B^* is weak*-compact.

In addition, prove that if $\mathcal{V}_1$ and $\mathcal{V}_2$ are normed spaces and $T \in \mathcal{B}(\mathcal{V}_1, \mathcal{V}_2)$, then

$$\|T\| = \sup \{|\Lambda(Tx)| \,|\, x \in \mathcal{V}_1,\, \Lambda \in (\mathcal{V}_2)^*,\, \|x\| \leq 1,\, \|\Lambda\| \leq 1\} .$$

Exercise 3

(a) Suppose $\mathcal{T}_1$ is a Hausdorff topology on a set X, and $\mathcal{T}_2$ is a topology on X under which X is compact. Show that if $\mathcal{T}_1 \subset \mathcal{T}_2$ then $\mathcal{T}_1 = \mathcal{T}_2$. (Hint: Consider closed sets.)

(b) Consider a set X with a topology $\mathcal{T}_X$ under which X is compact. Suppose there is a sequence $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}_+$, of continuous functions such that it separates points on X (meaning that for any $x_1, x_2 \in X$ with $x_1 \neq x_2$ there is n such that $f_n(x_1) \neq f_n(x_2)$). Set $c_n := 1 + \sup_{x \in X} |f_n(x)|$ and show that the formula

$$d(x, y) := \sum_{n=1}^{\infty} 2^{-n} \frac{1}{c_n} |f_n(x) - f_n(y)| , \qquad x, y \in X ,$$

defines a metric on X which induces $\mathcal{T}_X$. (Thus $\mathcal{T}_X$ is then metrizable.)

(Please turn over!)

Exercise 4

Consider the fermionic Fock space $\mathcal{F}^{(-)}$ generated by a one-particle Hilbert space $\mathfrak{h}$, and the corresponding creation and annihilation operators $a^*_-(g)$, and $a_-(g)$, $g \in \mathfrak{h}$, as defined in Exercise 6.10. Let also $\mathcal{F}$ denote the full Fock space, with a vacuum vector Ω, and $P^{(-)}$ the orthogonal projection onto $\mathcal{F}^{(-)}$. Prove the following statements.

(a) Consider some $N \in \mathbb{N}_+$ and $f_i \in \mathfrak{h}$, $i = 1, \dots, N$, and define the vector $\Psi \in \mathcal{F}$ by $\Psi_N = f_1 \otimes \cdots \otimes f_N$ and $\Psi_n = 0$ for $n \neq N$. Show that then

$$P^{(-)}\Psi = (N!)^{-1/2} a^*_-(f_1) \cdots a^*_-(f_N)\Omega \,.$$

(b) Show that for all $f, g \in \mathfrak{h}$, the *canonical anticommutation relations* hold:

$$a^*_-(f)a^*_-(g) = -a^*_-(g)a^*_-(f)\,, \quad a_-(f)a_-(g) = -a_-(g)a_-(f)\,, \quad \text{and}$$
$$a_-(f)a^*_-(g) = -a^*_-(g)a_-(f) + (f,g)_{\mathfrak{h}}1\,.$$

(c) Let $(e_i)_{i\in I}$ be an orthonormal basis of $\mathfrak{h}$. Show that the vectors $a^*_-(e_{i_1}) \cdots a^*_-(e_{i_N})\Omega$, with $N \in \mathbb{N}_0$, $i_n \in I$ for $n = 1, \dots, N$, and with all i_n different from each other, form an orthonormal basis of $\mathcal{F}^{(-)}$ *provided* one takes only one representative per class of sequences $(i_1, \dots, i_N)$ which differ only by a permutation.

Exercise 5

Consider next the special case $\mathfrak{h} = \ell_2(\mathbb{Z})$, and denote $a_-(x)$, $x \in \mathbb{Z}$, for $a_-(e_x)$ where $e_x(y) = 1$ for $y = x$ and zero otherwise. Similarly as in Exercise 6.10, define dense subspaces $D_0 \subset \mathcal{F}$ and $D_- \subset \mathcal{F}^{(-)}$ by $D_0 := \{\Psi \in \mathcal{F} \,|\, \sum_{N=0}^\infty N^4 \|\Psi_N\|^2 < \infty\}$ and $D_- := D_0 \cap \mathcal{F}^{(-)}$. Fix some $v_1 \in \ell_1(\mathbb{Z})$ and $v_2 \in \ell_1(\mathbb{Z} \times \mathbb{Z})$.

(a) Show that the following formulae define two operators B_1 and B_2 with a domain D_0 on $\mathcal{F}$. Set

$$(B_1\Psi)_N(x_1, \dots, x_N) := \sum_{i=1}^N v_1(x_i)\Psi_N(x)\,, \quad \Psi \in D_0,\ N \geq 1\,,$$
$$(B_2\Psi)_N(x_1, \dots, x_N) := \sum_{i,j=1;\, i\neq j}^N v_2(x_i, x_j)\Psi_N(x)\,, \quad \Psi \in D_0,\ N \geq 2\,,$$

and define $(B_1\Psi)_0 = 0 = (B_2\Psi)_0 = (B_2\Psi)_1$. (Both operators are actually multiplication operators; B_1 is called a one-body potential and B_2 a two-body potential.)

(b) Show that the operator $\sum_{x\in\mathbb{Z}} v_1(x)a^*_-(x)a_-(x)$ is the unique bounded extension of $P^{(-)}B_1\big|_{D_-}$ to an element in $\mathcal{B}(\mathcal{F}^{(-)})$.

(c) Show that the operator

$$\sum_{x,y\in\mathbb{Z}} v_2(x,y)a^*_-(x)a^*_-(y)a_-(y)a_-(x)$$

is the unique bounded extension of $P^{(-)}B_2\big|_{D_-}$ in $\mathcal{B}(\mathcal{F}^{(-)})$.

Note how easily the antisymmetrization can be handled by using the creation and annihilation operators here.