Introduction to Mathematical Physics: Spectral Theory — Homework set 7, 12. Nov 2010

There will be no lecture on Thursday 11.11. However, there will be exercise sessions on week 45: if needed, the problems of Exercise 6 will be completed instead of the lecture on Tuesday 9.11., and Exercise 7 will be discussed as usual on Friday 12.11. **Exercise sheet 8 will be distributed on Friday 12.11.**

Exercise 1

Consider a Hilbert space $\mathcal{H}$. Show that $\mathcal{H}$ with the "weak topology" as defined in item a) on p. 41 of the lecture notes is equal to $\mathcal{H}_{\mathrm{w}}$ defined in 13.4.

Exercise 2

Let $\mathcal{V}$ be a topological vector space for which $\mathcal{V}^*$ separates points on $\mathcal{V}$. Prove that the following statements hold for the weak topology on $\mathcal{V}$.

(a) If $x_n \in \mathcal{V}$, $n \in \mathbb{N}_+$, then $x_n \to 0$ weakly if and only if $\Lambda x_n \to 0$ for every $\Lambda \in \mathcal{V}^*$.

(b) Every originally convergent sequence is weakly convergent.

(c) $E \subset \mathcal{V}$ is weakly bounded if and only if $\sup_{x \in E} |\Lambda x| < \infty$ for every $\Lambda \in \mathcal{V}^*$.

(d) If $\dim \mathcal{V} = \infty$, then every weak neighborhood of 0 contains an infinite-dimensional subspace.

(e) Show that $\mathcal{V}_{\mathrm{w}}$ is not normable if $\dim \mathcal{V} = \infty$.

Exercise 3

Show that every *locally convex and locally bounded* topological vector space is *normable*. (Hint: Minkowski functional of a sufficiently nice neighborhood of zero. Together with Exercise 4.2.b this completes the proof of the statements made in section 4.6. of the lecture notes.)

(Please turn over!)

Exercise 4

Consider the spaces $\ell_p(\mathbb{N})$, $1 \le p < \infty$, which consist of sequences of complex numbers $z(n)$, $n \in \mathbb{N}$, for which $\|z\|_p < \infty$, with $\|z\|_p := (\sum_n |z(n)|^p)^{1/p}$. Define also $\ell_\infty(\mathbb{N})$ analogously using $\|z\|_\infty := \sup_n |z(n)|$. Since clearly $\ell_p(\mathbb{N}) = L^p(\mu)$, for μ equal to the counting measure on $\mathbb{N}$, every ℓ_p space is a Banach space.

(a) For $1 < p < \infty$ set $q := \frac{p}{p-1}$, and prove that $(\ell_p)^* = \ell_q$ in the following sense: there is a one-to-one correspondence between $\Lambda \in (\ell_p)^*$ and $w \in \ell_q$ determined by $\Lambda(z) = \sum_n w(n)z(n)$, for all $z \in \ell_p$. What happens for $p = 2$?

(b) Assume $1 < p < \infty$ and prove that there are sequences in ℓ_p which converge weakly but not in norm.

(c) In contrast, show that every weakly convergent sequence in ℓ_1 is also norm convergent. Do the weak and norm topologies coincide in this case?

Exercise 5

Prove that $(\ell_1)^* = \ell_\infty$, but $(\ell_\infty)^* \neq \ell_1$. (Hint for the second statement: Prove this first for real scalars using the Hahn-Banach theorem 12.3. Consider for instance the maps which take an average of the first n elements.)

Exercise 6

$\mathbf{L^p}$-spaces with $\mathbf{0 < p < 1}$

Fix $0 < p < 1$. Let $L^p = L^p([0,1])$ consist of those Lebesgue measurable functions f on the interval $[0,1]$ for which $\Delta(f) < \infty$, where

$$\Delta(f) := \int_0^1 \mathrm{d}t\, |f(t)|^p ,$$

with the usual identification of functions which are equal almost everywhere.

(a) Show that the formula $d(f,g) := \Delta(f-g)$ defines a translation invariant metric on L^p.

(b) Show that the corresponding metric topology makes L^p into a locally bounded F-space.

(c) Show that there are no other convex open sets beside the trivial ones, $\emptyset$ and L^p. (Hint: Consider an arbitrary $\varepsilon > 0$ and $f \in L^p$, and show that f lies in the convex hull of the open ball $B(0,\varepsilon)$. To do this, you can choose a large enough n and split the interval $[0,1]$ into n subintervals, each contributing equally to the integral defining $\Delta(f)$.)

(d) Show that $(L^p)^* = \{0\}$. What is the $(L^p)^*$-weak topology in this case?