Introduction to Mathematical Physics: Spectral Theory — Homework set 6, 5. Nov 2010

There will be no lectures nor exercises on weeks 42 and 43, and therefore you will have three weeks to finish these exercises. Next lecture is on Tuesday 2nd November.

Exercise 1

Prove Theorem 7.1 in the lecture notes. (Hint: The point of the exercise is to go through and fill in the details of the proof given in Rudin's Functional Analysis book.)

Exercise 2 (Absorbing sets)

Let X be a vector space. Then $A \subset X$ is called *absorbing* if for all $x \in X$ there is $t > 0$ for which $x \in tA$. Prove that if A is absorbing, then $0 \in A$, and that if $\mathcal{V}$ is a topological vector space and V is a neighborhood of 0, then V is absorbing.

Exercises 3 & 4 (Seminorms)

Let X be a vector space. A function $p : X \to \mathbb{R}$ is called a *seminorm* if it satisfies

(i) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,
(ii) $p(\alpha x) = |\alpha| p(x)$ for all $x \in X$ and $\alpha \in \mathbb{K}$.

Prove that then

(a) $p(0) = 0$ and $|p(x) - p(y)| \leq p(x - y)$ for all $x, y \in X$
(b) $p(x) \geq 0$ for all $x \in X$
(c) p is a norm if and only if $p(x) \neq 0$ for all $x \neq 0$
(d) $N := \{x \in X \,|\, p(x) = 0\}$ is a subspace of X, and there is a *unique norm* $\|\cdot\|$ on X/N for which $\|\pi_N(x)\| = p(x)$ for all $x \in X$
(e) The set $B := \{x \in X \,|\, p(x) < 1\}$ is convex, balanced, and absorbing.

Exercises 5 & 6 (Minkowski functional)

Let X be a vector space, and consider $A \subset X$ which is *convex and absorbing*. Then the map $\mu_A : X \to \mathbb{R}$ defined by

$$\mu_A(x) := \inf \left\{ t > 0 \,\middle|\, t^{-1} x \in A \right\}, \quad x \in X,$$

is called the *Minkowski functional* of A. (Since A is absorbing, $\mu_A(x)$ is always well-defined and non-negative.) Prove that then all of the following statements hold.

(a) $\mu_A(x + y) \leq \mu_A(x) + \mu_A(y)$ for all $x, y \in X$
(b) $\mu_A(tx) = t\mu_A(x)$ for all $x \in X$ and $t \geq 0$
(c) μ_A is a seminorm if A is also balanced
(d) Set $A_0 := \{x \in X \,|\, \mu_A(x) < 1\}$ and $A_1 := \{x \in X \,|\, \mu_A(x) \leq 1\}$. Show that then A_0 and A_1 are convex and absorbing, $A_0 \subset A \subset A_1$, and $\mu_{A_0} = \mu_A = \mu_{A_1}$.
(e) If p is a seminorm on X, then $p = \mu_B$ for the set B defined in item (e) in Exercise 3.

Exercise 7

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be F-spaces, with compatible invariant metrics d_1 and d_2, respectively. Show that $d((x_1, x_2), (y_1, y_2)) := d_1(x_1, y_1) + d_2(x_2, y_2)$ defines an invariant metric on $\mathcal{F}_1 \times \mathcal{F}_2$ which is compatible with its product topology. Show that $\mathcal{F}_1 \times \mathcal{F}_2$ is an F-space.

(Please turn over!)

Exercise 8

Let $N \in \mathbb{N}_+$ and assume $\mathcal{B}_n$, $n = 0, 1, \dots, N$ are Banach spaces. Show that a multilinear map $T : \prod_{n=1}^N \mathcal{B}_n \to \mathcal{B}_0$ is jointly continuous if and only if it is separately continuous, and that in this case there is $M \geq 0$ such that $\|T(x_1, x_2, \dots, x_N)\| \leq M \prod_{n=1}^N \|x_n\|$ for any choice of $x_n \in \mathcal{B}_n$, $n = 1, \dots, N$. (Hint: Induction in N.)

Exercise 9

(a) Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be topological vector spaces, and suppose that for each $n \in \mathbb{N}_+$ there is given a linear map $\Lambda_n : \mathcal{V}_1 \to \mathcal{V}_2$ such that the sequence (Λ_n) is equicontinuous. Define E as the collection of all points $x \in \mathcal{V}_1$ for which $(\Lambda_n x)$ forms a topological Cauchy sequence. Prove that E is a closed subspace of $\mathcal{V}_1$.

(b) Assume, in addition, that $\mathcal{V}_2$ is an F-space and that there is $D \subset \mathcal{V}_1$ such that D is dense, and $(\Lambda_n x)$ converges for every $x \in D$. Prove that then $\Lambda(x) := \lim_{n\to\infty} \Lambda_n x$ exists for *all* $x \in \mathcal{V}_1$, and that the map $\Lambda : \mathcal{V}_1 \to \mathcal{V}_2$ is continuous and linear.

Exercises 10 & 11

Let $\mathfrak{h}$ be a Hilbert space, and consider the standard Fock space generated by it: define $\mathcal{H}_0 = \mathbb{C}$, $\mathcal{H}_1 = \mathfrak{h}$, and $\mathcal{H}_N = \bigotimes_{n=1}^{N} \mathfrak{h}$, for $N = 2, 3, \dots$, and then set $\mathcal{F} := \bigoplus_{n=0}^{\infty} \mathcal{H}_N$. Consider some fixed $g \in \mathfrak{h}$.

(a) For $N \in \mathbb{N}_+$ prove that there is a unique continuous linear map $a_N : \mathcal{H}_N \to \mathcal{H}_{N-1}$ with

$$a_N\left(\otimes_{n=1}^N \psi_n\right) = \sqrt{N}(g, \psi_1)_{\mathfrak{h}} \otimes_{n=2}^N \psi_n\,, \qquad \text{for all } \psi \in \mathfrak{h}^N := \textstyle\prod_{n=1}^N \mathfrak{h}\,.$$

(Hint: Theorem 2.17 and Exercise 5.3. Recall that for any non-zero $f \in \mathfrak{h}$ there is an orthonormal basis of $\mathfrak{h}$ which contains $f/\|f\|$.)

(b) Show that $D_0 := \left\{\Psi \in \mathcal{F} \,\middle|\, \sum_{N=0}^\infty N\|\Psi_N\|^2 < \infty\right\}$ is a dense subspace of $\mathcal{F}$ which contains the vacuum vector $\Omega = (1, 0, 0, \dots)$.

(c) Prove that the equation $(a\Psi)_N = a_{N+1}\Psi_{N+1}$, $N = 0, 1, \dots$, defines an operator $D_0 \to \mathcal{F}$, and that this operator is *unbounded* if $g \neq 0$. Compute $a\Omega$.

(d) Show that there is a unique continuous linear map $c_N : \mathcal{H}_N \to \mathcal{H}_{N+1}$ with

$$c_N\left(\otimes_{n=1}^N \psi_n\right) = \sqrt{N+1}\, g \otimes \psi_1 \otimes \cdots \otimes \psi_N\,, \qquad \text{for all } \psi \in \mathfrak{h}^N\,,$$

for any choice of $N = 0, 1, \dots$. Prove that if we set $(c\Psi)_0 = 0$ and $(c\Psi)_N = c_{N-1}\Psi_{N-1}$, for $N \in \mathbb{N}_+$, then c is an operator $D_0 \to \mathcal{F}$ which is unbounded if $g \neq 0$. Compute $c\Omega$.

$a = a(g)$ is called the *annihilation operator* related to g on $\mathcal{F}$ and $c = c(g)$ is called the *creation operator* related to g.

Recall the fermionic Fock space defined after Proposition 2.21: $\mathcal{F}^{(-)} = \bigoplus_{N=0}^\infty \mathcal{H}_N^{(-)}$, where $\mathcal{H}_N^{(-)}$ is the totally antisymmetric subspace of $\mathcal{H}_N$. As before, let $P_N^{(-)}$ denote the orthogonal projection onto $\mathcal{H}_N^{(-)}$, and consider some fixed $g \in \mathfrak{h}$. The following statements show that the fermionic creation and annihilation operators, defined by restricting $a(g)$ and $c(g)$ to $\mathcal{F}^{(-)}$, are actually *bounded* operators.

(e) Show that the formulae $(P^{(-)}\Psi)_0 := \Psi_0$, $(P^{(-)}\Psi)_N := P_N^{(-)}\Psi_N$, for $N \in \mathbb{N}_+$, define an orthogonal projection $P^{(-)} : \mathcal{F} \to \mathcal{F}$ onto $\mathcal{F}^{(-)}$.

(f) Prove that $D_- := D_0 \cap \mathcal{F}^{(-)}$ is a dense subspace of $\mathcal{F}^{(-)}$, and consider the restrictions of $a(g)$ and $c(g)$ to $\mathcal{F}^{(-)}$, i.e., the maps $\tilde{a} := P^{(-)}a(g)|_{D_-}$ and $\tilde{c} := P^{(-)}c(g)|_{D_-}$. Show that there are unique $a_-(g), c_-(g) \in \mathcal{B}(\mathcal{F}^{(-)})$ such that $a_-(g)|_{D_-} = \tilde{a}$, $c_-(g)|_{D_-} = \tilde{c}$, and that then $\|a_-(g)\| = \|g\|_{\mathfrak{h}} = \|c_-(g)\|$. (Hint: What happens to $P_N^{(-)}(\otimes_{n=1}^N \psi_n)$, if $\psi_i = \psi_j$ for some $i \neq j$?)

(g) Show that $c_-(g)$ is the adjoint of $a_-(g)$. (In this context, usually denoted by $a_-^*(g)$.)