Introduction to Mathematical Physics: Spectral Theory | Homework set 5, Friday 15. Oct 2010

Exercise 1

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be normed spaces, and consider a linear map $\Lambda : \mathcal{V}_1 \to \mathcal{V}_2$. Show that the previous and the new definition of Λ being "bounded" coincide: show that $\Lambda(E)$ is bounded for every bounded $E \subset \mathcal{V}_1$ if and only if $\|\Lambda\| := \sup\{\|\Lambda x\| \,|\, x \in \mathcal{V}_1, \|x\| = 1\} < \infty$. (By Theorem 8.3 this proves that $\mathcal{B}(\mathcal{H})$ is equal to the collection of continuous linear transformations of $\mathcal{H}$.)

Exercise 2

(a) Suppose d_1 and d_2 are *invariant* metrics on a vector space X, and that they induce the same topology on X. Prove that then d_1 and d_2 have the same Cauchy sequences *and* that X is complete with respect to d_1 if and only if it is complete with respect to d_2.

(b) Find a metric d_2 on $\mathbb{R}$ such that it induces the standard topology, but for which $\mathbb{R}$ is *not* complete with respect to d_2. (Hint: The map $\phi(x) = x/(1 + |x|)$ might become handy.)

Exercise 3

Continuous extensions of densely defined linear maps:

(a) Let $\mathcal{V}$ and $\mathcal{F}$ be topological vector spaces, and assume that $\mathcal{F}$ is an F-space. Suppose M is a *dense* subspace of $\mathcal{V}$, and $\Lambda : M \to \mathcal{F}$ is *continuous and linear*. Show that Λ has a unique continuous extension to a linear map $\overline{\Lambda} : \mathcal{V} \to \mathcal{F}$. (Hint: Start by showing that there is a sequence $(V_n)_{n\in\mathbb{N}_+}$ of neighborhoods of zero in $\mathcal{V}$ such that each V_n is balanced, $V_n + V_n \subset V_{n-1}$, and $d(0, \Lambda x) < 2^{-n}$ for all $x \in M \cap V_n$. Then for any $x \in \mathcal{V}$ you can choose $x_n \in (x + V_n) \cap M$. Show that the sequence (Λx_n) is Cauchy and that the limit point is independent of the choice of x_n.)

(b) Let $\mathcal{H}$ be a Hilbert space, and D be its dense subspace. Suppose $A : D \to \mathcal{H}$ is linear and satisfies $\|A\| < \infty$. Show that there is a unique operator $\overline{A} \in \mathcal{B}(\mathcal{H})$ for which $\overline{A}\psi = A\psi$ for all $\psi \in D$, and that $\|\overline{A}\| = \|A\|$. This proves that every densely defined bounded operator on a Hilbert space has a unique extension in $\mathcal{B}(\mathcal{H})$. (Hint: (a).)

(Please turn over!)

Exercise 4

Assume that all of the following conditions hold:

(a) $\mathcal{V}$ and $\mathcal{V}'$ are topological vector spaces.

(b) $\Lambda : \mathcal{V} \to \mathcal{V}'$ is linear.

(c) N is a *closed* subspace of $\mathcal{V}$, and let $\pi : \mathcal{V} \to \mathcal{V}/N$ denote the related quotient map.

(d) $\Lambda x = 0$ for all $x \in N$.

Show that then there is unique $f : \mathcal{V}/N \to \mathcal{V}'$ satisfying $\Lambda = f \circ \pi$. Prove that this f is linear. Show that f is continuous if and only if Λ is continuous, and that f is open if and only if Λ is open. (This shows that it is fairly harmless to "divide out" the kernel of a continuous linear map between topological vector spaces.)

Exercise 5

Let $\mathcal{H}$ be a Hilbert space, and suppose N is its closed subspace. Show that there is a scalar product on $\mathcal{H}/N$, compatible with its quotient norm, such that $\mathcal{H}/N \cong N^{\perp}$. (This implies that $\mathcal{H}/N$ with its quotient topology is not only a Banach, but also a Hilbert space.)