Introduction to Mathematical Physics: Homework set 4
Spectral Theory Friday 8. Oct 2010

Exercise 1

Prove that, if A is a subset of a vector space, then $2A \subset A + A$. Find $A \subset \mathbb{R}$ such that $2A \neq A + A$.

Exercise 2

Let $\mathcal{V}$ be a normed space (endowed with its norm topology).

(a) Show that $A \subset \mathcal{V}$ is topologically bounded if and only if there is $M > 0$ such that $\|x\| \leq M$ for all $x \in A$.
(b) Show that $\mathcal{V}$ is a locally convex and locally bounded topological vector space.

Exercise 3

Let $\mathcal{H}$ be a Hilbert space. Show that $\mathcal{B}(\mathcal{H})$, endowed with its weak operator topology, is a topological vector space. (Hint: Sometimes it is best to go back to definitions. Note that you cannot assume that the topology is metrizable.)

Exercise 4

Let $\mathcal{V}$ be a topological vector space. For any $A \subset \mathcal{V}$, let $\overline{A}$ denote the closure of A, and A° denote the interior of A. Prove all of the following statements.

(a) If $A \subset \mathcal{V}$ is *convex*, then so are $\overline{A}$ and A°.
(b) If $A \subset \mathcal{V}$ is *balanced*, then so is $\overline{A}$. If, in addition, $0 \in A^\circ$, then A° is balanced.
(c) If $A \subset \mathcal{V}$ is *bounded*, then so is $\overline{A}$. (Hint: Corollary 4.9.)

(Please turn over!)

Exercise 5

Let X be a vector space. For any collection of $n \in \mathbb{N}_+$ vectors $x_i \in X$, $i = 1, 2, \ldots, n$, the vector $y \in X$ is a *convex combination* of (x_i) if there are $t_i \geq 0$, $i = 1, 2, \ldots, n$, such that

$$y = \sum_{i=1}^{n} t_i x_i, \quad \text{and} \quad \sum_{i=1}^{n} t_i = 1 \,.$$

The *convex hull* of $A \subset X$ is defined as

$$\mathrm{Hull}(A) := \{ y \in X \mid y \text{ is a convex combination of vectors in } A \} \,.$$

Here y is a convex combination with $n = 1$ if and only if $y \in A$. Thus always $A \subset \mathrm{Hull}(A)$.

(a) What is $\mathrm{Hull}(A) \subset \mathbb{R}^2$, for $A = \{0, \hat{e}_1, \hat{e}_2\}$ and for $A = \{0, \hat{e}_1, 2\hat{e}_1\}$? (Try to depict the sets.)

(b) Prove that $\mathrm{Hull}(A)$ is convex, and that it is equal to the intersection of all convex sets which contain A.

(c) Assume $A \subset \mathcal{V}$, where $\mathcal{V}$ is a topological vector space and A is open. Show that $\mathrm{Hull}(A)$ is open.

(d) Assume $A \subset \mathcal{V}$, where $\mathcal{V}$ is a *locally convex* topological vector space and A is bounded. Show that $\mathrm{Hull}(A)$ is bounded.