Introduction to Mathematical Physics: Spectral Theory — Homework set 3

Friday 1. Oct 2010

General note about the exercises: even if you have not finished proving some exercise, it is OK to use its results in the problems following it.

Exercise 1

Let $T, S \in \mathcal{B}(\mathcal{H})$, and $\alpha \in \mathbb{C}$ be arbitrary. Show that all of the following statements hold for the related adjoint operators.

(a) $(T+S)^* = T^* + S^*$
(b) $(\alpha T)^* = \alpha^* T^*$
(c) $(ST)^* = T^* S^*$ (notation: $ST = S \circ T$)
(d) $T^{**} = T$ (notation: $T^{**} = (T^*)^*$)
(e) $\|T^* T\| = \|T\|^2$

(This proves that * is an involution on $\mathcal{B}(\mathcal{H})$ which makes it into a C^*-algebra.)

Exercise 2

Let $M \subset \mathcal{H}$ be a subspace (not necessarily closed). Show that $(M^\perp)^\perp = \overline{M}$.

Exercise 3

Consider an arbitrary $T \in \mathcal{B}(\mathcal{H})$.

(a) Prove that the "expectation values" of T determine it uniquely: show that, if $(\psi, T\psi) = 0$ for all $\psi \in \mathcal{H}$, then $T = 0$. (Hint: generalize the polarization identity.)
(b) Prove that $\mathrm{Ker}(T^*) = R(T)^\perp$ and $\mathrm{Ker}(T) = R(T^*)^\perp$.

Exercise 4

Suppose $P \in \mathcal{B}(\mathcal{H})$ is a non-zero projection. Show that the following statements are equivalent:

(a) P is an orthogonal projection.
(b) P is self-adjoint.
(c) P is non-negative (meaning that $(\psi, P\psi) \geq 0$ for all $\psi \in \mathcal{H}$).
(d) P is normal.
(e) $\|P\| = 1$.
(f) $\|P\| \leq 1$.

(Please turn over!)

Exercise 5

Vector and operator *sequences*: types of convergence

By relying on the definitions given in Section 3.5 of the lecture notes, first prove the following statements about convergence of sequences in these topologies.

(a) Consider a sequence $(\psi_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$. Then ψ_n converges to $\psi \in \mathcal{H}$

in norm if $\|\psi - \psi_n\| \to 0$ when $n \to \infty$.
weakly if $(\phi, \psi - \psi_n) \overset{n\to\infty}{\longrightarrow} 0$ for all $\phi \in \mathcal{H}$.

(b) For a sequence of bounded operators, $(T_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(\mathcal{H})$, there are three possibilities: T_n converges to $T \in \mathcal{B}(\mathcal{H})$

uniformly, or **in norm**, if $\|T - T_n\|_{\mathcal{B}(\mathcal{H})} \to 0$ when $n \to \infty$.
strongly if $\|T\psi - T_n\psi\|_{\mathcal{H}} \overset{n\to\infty}{\longrightarrow} 0$ for all $\psi \in \mathcal{H}$.
weakly if $(\phi, T\psi - T_n\psi) \overset{n\to\infty}{\longrightarrow} 0$ for all $\phi, \psi \in \mathcal{H}$.

We will use the following notations for the various types of convergence: lim or $\to$ for norm-convergence, s-lim or $\overset{s}{\to}$ for strong convergence, and w-lim or $\overset{w}{\to}$ for weak convergence.

Show that

(c) $\psi_n \to \psi \implies \psi_n \overset{w}{\to} \psi$
(d) $T_n \to T \implies T_n \overset{s}{\to} T \implies T_n \overset{w}{\to} T$

Exercise 6

Let $\mathcal{H} = \ell_2(\mathbb{N})$. (That is, let $\mathcal{H} = L^2(\mu)$ with μ the counting measure on $\mathbb{N} = \{0, 1, \ldots\}$.) Consider the following mappings defined for $\psi \in \mathcal{H}$, $n \geq 1$, and $k \in \mathbb{N}$:

$$(T_n^{(1)}\psi)(k) = \frac{1}{n}\psi(k), \tag{1}$$

$$(T_n^{(2)}\psi)(k) = \begin{cases} \psi(k), & \text{if } k \geq n, \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

$$(T_n^{(3)}\psi)(k) = \begin{cases} \psi(k-n), & \text{if } k \geq n, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

In other words, $T_n^{(1)}$ is division by n, $T_n^{(2)}$ is cancellation of the first n elements, and $T_n^{(3)}$ is right-shift by n.

Prove that

(a) $T_n^{(i)} \in \mathcal{B}(\mathcal{H})$ for $i = 1, 2, 3$ and $n \geq 1$.
(b) $T_n^{(1)} \to 0$ in norm.
(c) $T_n^{(2)} \overset{s}{\to} 0$ but $T_n^{(2)} \not\to 0$.
(d) $T_n^{(3)} \overset{w}{\to} 0$ but $T_n^{(3)} \overset{s}{\not\to} 0$ and $T_n^{(3)} \not\to 0$.