Introduction to Mathematical Physics: Spectral Theory — Homework set 2, Friday 24. Sep 2010

In all of the exercises, $\mathcal{H}$ denotes a *complex* Hilbert space with a scalar product $(\cdot,\cdot)$.

Exercise 1

Prove that the followings maps are continuous as maps from $\mathcal{H}$ to $\mathbb{C}$

(a) $\psi \mapsto \|\psi\|$

(b) For any fixed $\phi \in \mathcal{H}$, $\psi \mapsto (\phi, \psi)$

Exercise 2

Polarization identity

Suppose $\mathcal{V}$ is a complex vector space with a scalar product $(\cdot,\cdot)$ and a norm $\|\psi\| = \sqrt{(\psi,\psi)}$ (that is, like a Hilbert space, but not necessarily complete). Prove that for all $\phi, \psi \in \mathcal{V}$,

$$(\phi,\psi) = \frac{1}{4}\left(\|\phi+\psi\|^2 - \|\phi-\psi\|^2 - \mathrm{i}\|\phi+\mathrm{i}\psi\|^2 + \mathrm{i}\|\phi-\mathrm{i}\psi\|^2\right) .$$

Exercise 3

Show that $\mathbb{C}^{2\times 2}$, endowed with the matrix norm, is not a Hilbert space (*i.e.*, that there exists no scalar product which would yield the norm). (Hint: This can be a very easy exercise.)

Exercise 4

Let $I \neq \emptyset$ be an index set for a family $\mathcal{H}_i$, $i \in I$, of Hilbert spaces. (Note: I does not need to be countable.)

(a) Show that $\bigoplus_{i\in I} \mathcal{H}_i$, as defined in the lectures notes, is a Hilbert space.

(b) Let $\mathcal{V} = \{(\psi_i) \in \prod_{i\in I} \mathcal{H}_i \,|\, \psi_i \neq 0 \text{ only for finitely many } i\}$, and define addition and scalar multiplication componentwise. Show that $\mathcal{V}$ is a vector space, and that the equation $((\phi_i),(\psi_i))_{\mathcal{V}} := \sum_{i\in I}(\phi_i,\psi_i)_{\mathcal{H}_i}$ defines a scalar product on $\mathcal{V}$.

(c) Show that the Hilbert space completion (see Section 2.10) of $\mathcal{V}$ is $\bigoplus_{i\in I} \mathcal{H}_i$.

Exercise 5

Two Hilbert spaces are said to be isomorphic, if there exists a *unitary map* between them (a map is unitary if it is linear, invertible, and preserves the scalar product). We denote this by $\mathcal{H}_1 \cong \mathcal{H}_2$. Show that

(a) $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \cong L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ (a particle with two "spin"-components)

(b) $L^2([0,1]^d) \otimes L^2([0,1]^{d'}) \cong L^2([0,1]^{d+d'})$ for any d, $d' \in \mathbb{N}_+$ (Hint: Fourier series.)

Exercise 6

Prove Proposition 2.21 in the lecture notes. (Properties of $\mathcal{H}_N^{(\pm)}$, and $P_N^{(\pm)}$, the bosonic and fermionic subspaces of $\otimes^N \mathcal{H}_1$ and their orthogonal projections.)