Introduction to Mathematical Physics: Spectral Theory — Homework set 1

Friday 17. Sep 2010

Exercise 1

Let $J \in \mathbb{C}^{d\times d}$ be a simple "Jordan block" matrix, i.e., define $J_{ij} = \mathbb{1}(j = i+1)$ for all $i, j = 1, 2, \ldots, d$. Consider an arbitrary integer $n \geq 0$, and show that

(a) $(J^n)_{ij} = \mathbb{1}(j = i+n)$, for all $i, j = 1, 2, \ldots, d$

(b) $\|J^n\| = \mathbb{1}(n < d)$

Exercise 2

Consider a block matrix $M = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \in \mathbb{C}^{d\times d}$, where $B_1 \in C^{n\times n}$ and $B_2 \in C^{(d-n)\times(d-n)}$ for some $1 \leq n < d$. Show that $\|M\| = \max(\|B_1\|, \|B_2\|)$.

Exercise 3

Let $M \in \mathbb{C}^{d\times d}$. Show that the sum in

$$\mathrm{e}^M := \sum_{n=0}^{\infty} \frac{1}{n!} M^n$$

is absolutely convergent and $\|\mathrm{e}^M\| \leq \mathrm{e}^{\|M\|}$. (Reminder: $M^0 = 1$ in such sums.)

Exercise 4

Assume $A, B \in \mathbb{C}^{d\times d}$ *commute*, $AB = BA$. Show that then

(a) $(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^{n-k} B^k$, for any $n \in \mathbb{N}_+$ (here $\binom{n}{k} = \frac{n!}{k!(n-k)!}$)

(b) $\mathrm{e}^{A+B} = \mathrm{e}^A \mathrm{e}^B$

Exercise 5

Consider a map $t \mapsto M(t)$ from an interval $(t_1, t_2) \subset \mathbb{R}$ to $\mathbb{C}^{d\times d}$. Suppose $t_0 \in (t_1, t_2)$ is such that $M(t)$ is differentiable at t_0 (with derivative $M'(t_0)$), $M(t)$ is invertible for all t, and $t \mapsto M(t)^{-1}$ is continuous at t_0. Show that then

$$\frac{\mathrm{d}}{\mathrm{d}t} M(t)^{-1} \Big|_{t=t_0} = -M(t_0)^{-1} M'(t_0) M(t_0)^{-1} .$$

Exercise 6

Consider an arbitrary $M \in \mathbb{C}^{d\times d}$. Suppose that in its Jordan normal form decomposition, its eigenvalues are $\lambda \in \sigma(M)$ and it has $N \geq 1$ blocks with block sizes d_n, $n = 1, \ldots, N$. Define $\alpha = \max_{\lambda\in\sigma(M)}(\mathrm{Re}\,\lambda)$ and $\bar{d} = \max_n d_n$. Show that there is $C \geq 1$ such that for all $t \geq 0$

(a) $\|\mathrm{e}^{tM}\| \leq C\mathrm{e}^{t(\alpha+1)}$

(b) $\|\mathrm{e}^{tM}\| \leq C(1+t)^{\bar{d}-1}\mathrm{e}^{t\alpha}$

(If $\alpha < 0$, these bounds can be much better than the one following from Exercise 3.)