Introduction to Mathematical Physics: Spectral Theory Homework set 12*
(20. Dec 2010)

This is a bonus exercise set: you can get up to 2 points per problem, and these will be added to whatever points have been accumulated over the semester. Your written solutions should be returned directly to the lecturer, preferably by Monday 20.12. (If you wish to return your solutions later than that, please send e-mail to the lecturer.)

On **week 50**, there will be a lecture on Tuesday 14.12. On Thursday 16.12. we will have a discussion session where you can ask questions, and possibly some bonus material to the course. The exam is on 17.12., as announced earlier.

Exercise 1

Suppose $\mathcal{H}$ is a Hilbert space, and that $T \in \mathcal{B}(\mathcal{H})$. Prove that $(\psi, T\psi) \geq 0$ for all $\psi \in \mathcal{H}$ if and only if $T^* = T$ and $\sigma(T) \subset [0, \infty)$. (This shows that an operator T is positive if and only if it is a positive element of the C^*-algebra $\mathcal{B}(\mathcal{H})$.)

Exercise 2

Suppose $\mathcal{H}$ is a Hilbert space, and that $A \in \mathcal{B}(\mathcal{H})$ is *self-adjoint*. Let E denote the spectral decomposition of A. For $\varepsilon, a, b \in \mathbb{R}$, which satisfy $a < b$ and $\varepsilon > 0$ define

$$P(\varepsilon) := \frac{1}{2\pi \mathrm{i}} \int_a^b \mathrm{d}t \, \left[((t - \mathrm{i}\varepsilon)1 - A)^{-1} - ((t + \mathrm{i}\varepsilon)1 - A)^{-1} \right] .$$

(a) Explain why $P(\varepsilon)$ is well defined as a vector valued integral in $\mathcal{B}(\mathcal{H})$. Show that there is $f_\varepsilon \in L^\infty(E)$ for which $P(\varepsilon) = f_\varepsilon(A)$, in the sense of symbolic calculus.

(b) Suppose ε_n, $n \in \mathbb{N}_+$, is a sequence in $(0, \infty)$ for which $\lim_{n\to\infty} \varepsilon_n = 0$. Show that then $P(\varepsilon_n) \to \frac{1}{2} [E([a, b]) + E((a, b))]$ as $n \to \infty$ in the *strong operator topology*. (See Exercise 3.5.)

Exercise 3

Suppose $\mathcal{H}$ is a Hilbert space, and that $T \in \mathcal{B}(\mathcal{H})$. $\lambda \in \mathbb{C}$ is an *eigenvalue* of T if there is $\psi \in \mathcal{H}$, $\psi \neq 0$, such that $T\psi = \lambda\psi$. The collection of all eigenvalues of T is called the *pure point spectrum*, denoted by $\sigma_{\mathrm{pp}}(T)$.

Consider the Hilbert space $\mathcal{H} = \ell_2(\mathbb{N})$, as in Exercise 3.6. For $n \in \mathbb{N}$, $\psi \in \mathcal{H}$, define $(S_{\mathrm{L}}\psi)(n) := \psi(n + 1)$ and set $(S_{\mathrm{R}}\psi)(n) := \psi(n - 1)$, if $n \geq 1$, and $(S_{\mathrm{R}}\psi)(0) := 0$.

(a) Show that $S_{\mathrm{L}}, S_{\mathrm{R}} \in \mathcal{B}(\mathcal{H})$, and $(S_{\mathrm{R}})^* = S_{\mathrm{L}}$.

(b) Prove that $S_{\mathrm{L}}S_{\mathrm{R}} = 1 \neq S_{\mathrm{R}}S_{\mathrm{L}}$, and conclude that S_{R} and S_{L} are not normal.

(c) Prove that $\sigma(S_{\mathrm{L}}) = \{\lambda \in \mathbb{C} \,|\, |\lambda| \leq 1\} = \sigma(S_{\mathrm{R}})$.

(d) Show that $\sigma_{\mathrm{pp}}(S_{\mathrm{R}}) = \emptyset$ while $\sigma_{\mathrm{pp}}(S_{\mathrm{L}}) = \{\lambda \in \mathbb{C} \,|\, |\lambda| < 1\}$.

(Please turn over!)

Exercise 4

Let $\mathcal{H}$ be a Hilbert space. Assume that $T \in \mathcal{B}(\mathcal{H})$ is *normal*, and let E denote its spectral decomposition.

(a) Suppose $\lambda_0 \in \sigma(T)$. Show that λ_0 is an eigenvalue of T if and only if $E(\{\lambda_0\}) \neq 0$.

(b) Show that every isolated point of $\sigma(T)$ is an eigenvalue.

(c) Show that if the set $\sigma(T)$ is *countable*, then for every $\psi \in \mathcal{H}$ there corresponds a sequence $\psi_\lambda \in \mathcal{H}$, $\lambda \in \sigma(T)$, such that

1. Each ψ_λ satisfies $T\psi_\lambda = \lambda\psi_\lambda$. (Thus it is either an eigenvector or zero.)
2. $\psi_\lambda \perp \psi_{\lambda'}$ whenever $\lambda \neq \lambda'$.
3. $\psi = \sum_{\lambda\in\sigma(T)} \psi_\lambda$.

(d) Show that, if $\mathcal{H}$ is separable, then $\sigma_{\mathrm{pp}}(T)$ is at most countable.

(This proves that the pathologies in Exercises 3 and 5 cannot happen for normal operators.)

Exercise 5

Consider the Hilbert space $\mathcal{H} = \ell_2(\mathbb{N})$ and the operator $S_{\mathrm{R}} \in \mathcal{B}(\mathcal{H})$, as in Exercise 3. Suppose $v_n \in \mathbb{C} \setminus \{0\}$, $n \in \mathbb{N}$, are such that $v_n \to 0$ when $n \to \infty$, and define $(V\psi)(n) := v_n\psi(n)$ for $n \in \mathbb{N}$, $\psi \in \mathcal{H}$. (V is then a multiplication operator.)

(a) Show that $V \in \mathcal{B}(\mathcal{H})$. Thus $T := VS_{\mathrm{R}}$ is also a bounded operator.

(b) Compute $\|T^n\|$ for $n \in \mathbb{N}_+$.

(c) Show that $\sigma(T) = \{0\}$.

(d) Show that $\sigma_{\mathrm{pp}}(T) = \emptyset$. (Thus it is possible that an isolated point in the spectrum of a non-normal operator does not belong to the pure point spectrum of the operator.)

(e) Show that the resolvent $R_z := (z1-T)^{-1}$ of T does not have a pole at $z = 0$. (Definition in Exercise 11.6.)

Exercise 6

Complete the proof of the Jordan decomposition theorem, as stated on p. 12 of the lecture notes. (Hint: use the results in the homework set 11.)