Introduction to Mathematical Physics: Spectral Theory

Homework set 11
10. Dec 2010

The exam of the course will be held in the first general examination session of the department, on **Friday 17.12. between 8:30–12:30 in the Exactum auditorium A111 (or B123)**. If you wish to take the exam, *please register in WebOodi by 9.12. the latest.* After this, you can register to the exam only by e-mail.

Exercise 1

(a) Suppose J is a *proper* ideal in a commutative complex algebra $\mathcal{C}$. Show that if $x \in \mathcal{C}$ is invertible, then $x \notin J$.

(b) Suppose J is an ideal in a commutative *Banach algebra* $\mathcal{C}$. Show that its closure $\overline{J}$ is also an ideal.

Exercise 2

Suppose $\mathcal{A}$ is a Banach algebra whose unit is e, and $E \subset \mathcal{A}$. Define

$$F := \{e\} \cup \{x_1 x_2 \cdots x_n \,|\, n \in \mathbb{N}_+ \text{ and } x_i \in E, \text{ for all } i = 1, 2, \ldots, n\} \, .$$

Let M be the closure of the vector space spanned by F. Show that M is a Banach algebra; it is called the Banach subalgebra generated by E. Show also that, if $xy = yx$ for every $x, y \in E$, then M is commutative.

Exercise 3

Suppose $\mathcal{A}$ is a Banach algebra whose unit is e, and $x \in \mathcal{A}$. Denote the spectrum of x by $\sigma(x)$, the resolvent set of x by $\rho(x)$, and the resolvent map by $R_z := (ze - x)^{-1}$, with $z \in \rho(x)$. Prove that for all $z, z' \in \rho(x)$,

(a) $R_{z'} - R_z = -(z' - z) R_{z'} R_z$,

(b) $R_{z'}$ commutes with R_z,

(c) R_z commutes with x.

(Please turn over!)

Exercise 4

Continuation from the previous exercise:

Let γ_r denote the positively oriented circle of radius r around 0: set $\gamma_r(t) := r\mathrm{e}^{\mathrm{i}t}$ for $t \in [0, 2\pi]$. Assume $r > \|x\|$ and prove that as a vector valued integral

$$\frac{1}{2\pi\mathrm{i}} \oint_{\gamma_r} \mathrm{d}z\, R_z = e\,.$$

Suppose then that $z_0 \in \rho(x)$ and let γ be a path in $\rho(x) \setminus \{z_0\}$ which goes once around $\sigma(x)$ but does not go around z_0. (Meaning that $\mathrm{Ind}_\gamma(z_0) = 0$ and $\mathrm{Ind}_\gamma(\lambda) = 1$ for all $\lambda \in \sigma(x)$. Note that if z_0 belongs to a bounded component of $\rho(x)$, the "path" will actually be a union of two paths, and should be called a *cycle*. An integral over a cycle is defined as the sum of the integrals over its component paths.) Show that for all $n \in \mathbb{Z}$, as vector valued integrals

$$\frac{1}{2\pi\mathrm{i}} \oint_{\gamma} \mathrm{d}z\, (z_0 - z)^n R_z = (z_0 e - x)^n\,.$$

(Hint: Induction from $n = 0$.)

Exercise 5

Continuation from Exercise 3:

Assume then that λ is an isolated point in the spectrum of x, *i.e.*, that $\lambda \in \sigma(x)$ and there is some $\varepsilon > 0$ so that $\sigma(x) \cap B(\lambda, 2\varepsilon)$ contains only λ. Let γ denote the positively oriented circle of radius ε around λ: set $\gamma(t) := \lambda + \varepsilon\mathrm{e}^{\mathrm{i}t}$ for $t \in [0, 2\pi]$. Define $P, T \in \mathcal{A}$ by

$$P := \frac{1}{2\pi\mathrm{i}} \oint_{\gamma} \mathrm{d}z\, R_z\,, \quad T := \frac{1}{2\pi\mathrm{i}} \oint_{\gamma} \mathrm{d}z\, (z - \lambda) R_z\,.$$

(a) Show that the definitions make sense as vector valued integrals.
(b) Show that P, T, and x commute with each other. (Hint: Theorems 15.3.d and 16.2.)
(c) Show that $T + \lambda P = xP$.
(d) Prove that P is *idempotent*: $P^2 = P$. (Hint: Cauchy's theorem.)

Exercise 6

Continuation from the previous exercise:

(a) Show that $\lim_{z\to\lambda} \|R_z\| = \infty$.

Thus R_z is singular at λ. Suppose now that there is $n \in \mathbb{N}_+$ such that $z \mapsto |z - \lambda|^n \|R_z\|$ is bounded in some neighborhood of λ. Let $m \in \mathbb{N}_+$ be the smallest of such n.

(b) Show that then λ is a *pole of order* m of R_z: show that there are $c_n \in \mathcal{A}$, $n = 0, 1, \ldots, m$, such that $c_m \neq 0$ and the map $f : \rho(x) \cup \{\lambda\} \to \mathcal{A}$ defined by $f(\lambda) := c_0$ and $f(z) := R_z - \sum_{n=1}^m (z - \lambda)^{-n} c_n$, $z \in \rho(x)$, is holomorphic. (Hint: Show that $(z - \lambda)^{m+1} R_z$ is then holomorphic on $\rho(x) \cup \{\lambda\}$, and guess from this integral expressions for c_n.)
(c) Show that $c_n = (x - \lambda e)^{n-1} P$ for any $n = 1, \ldots, m$.
(d) Show that T is *nilpotent*: show that $T^m = 0$. (Hint: $c_2 = T$ if $m \geq 2$.)