**Introduction to Mathematical Physics: Spectral Theory** — Homework set 10, 3. Dec 2010

The exam is planned to be held in the first general examination session of the department, which is on **Friday 17.12. between 8:30–12:30**. *If you need the study credits from the course, but cannot come to this exam, please send e-mail to the lecturer as soon as possible.* The exact date will be fixed on Thursday 2.12.

## Exercise 1

(a) Consider Banach spaces $\mathcal{B}_1$, $\mathcal{B}_2$, and an open subset $\Omega \subset \mathcal{B}_1$. Suppose $f : \Omega \to \mathcal{B}_2$ is Fréchet differentiable at $a \in \Omega$, with a Fréchet derivative $T_a \in \mathcal{B}(\mathcal{B}_1, \mathcal{B}_2)$. Show that $f$ is continuous at $a$ and that the Fréchet derivative is unique.

(b) Assume $\Omega \subset \mathbb{C}$ is open, $\mathcal{B}$ is a complex Banach space, and $f : \Omega \to \mathcal{B}$ is strongly holomorphic. Let $f'$ denote the derivative of $f$. Consider some $a \in \Omega$ and define $T_a : \mathbb{C} \to \mathcal{B}$ by $T_a(\alpha) := \alpha f'(a)$. Show that $T_a$ is the Fréchet derivative of $f$ at $a$.

## Exercise 2

Assume $K$ is a nonempty compact Hausdorff space, and let $C(K)$ denote the collection of continuous functions $K \to \mathbb{C}$, with the standard definition of addition and scalar multiplication. For $f \in C(K)$ let $\|f\| := \sup_{x\in K} |f(x)|$, and define multiplication by $(fg)(x) := f(x)g(x)$ for $f, g \in C(K)$. Show that these definitions make $C(K)$ into a Banach algebra which is *commutative*: $fg = gf$ for all $f, g \in C(K)$.

## Exercise 3

**Fundamental theorem of calculus for vector valued integrals:**

Suppose $\mathcal{B}$ is a Banach space and $f : [0, T] \to \mathcal{B}$ is continuous, for some $T > 0$. By 15.3., we can then define $F : [0, T] \to \mathcal{B}$ through the vector valued integral

$$F(t) := \int_0^t \mathrm{d}s\, f(s)\,, \quad t \in [0, T]\,.$$

Show that $F$ is continuous, and that for any $t \in (0, T)$

$$F'(t) := \lim_{s\to t} \frac{1}{t-s}(F(t) - F(s)) = f(t)\,.$$

(Thus $F$ is continuously differentiable.)

(Please turn over!)

## Exercise 4

Assume $\Omega \subset \mathbb{C}$ is open, $\mathcal{F}$ is a complex Fréchet space, and $f : \Omega \to \mathcal{F}$ is strongly holomorphic. Let $f' : \Omega \to \mathcal{F}$ denote the derivative of $f$.

(a) Show that $f'$ is strongly holomorphic on $\Omega$.

(b) Therefore, we can iteratively define strongly holomorphic functions $f^{(n)} : \Omega \to \mathcal{F}$, $n = 0, 1, \ldots$, by the formula $f^{(n)}(z) := \frac{\mathrm{d}}{\mathrm{d}z} f^{(n-1)}(z)$, $n \in \mathbb{N}_+$, with $f^{(0)} := f$. Show that $\Lambda \circ f^{(n)} = \frac{\mathrm{d}^n}{\mathrm{d}^n z}(\Lambda \circ f)$ for any $\Lambda \in \mathcal{F}^*$ and $n \in \mathbb{N}_+$.

(c) Consider some $z_0 \in \Omega$, and the closed path $\gamma : [0, 2\pi] \to \Omega$ defined by $\gamma(t) := z_0 + \varepsilon \mathrm{e}^{\mathrm{i}t}$ for such a small $\varepsilon > 0$ that the closed ball $\overline{B(z_0, \varepsilon)} \subset \Omega$. (In other words, $\gamma$ is a positively oriented circle around $z_0$ with so small radius that the corresponding closed disc belongs to $\Omega$.) Show that for any $n = 0, 1, \ldots$,

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \oint_\gamma \mathrm{d}\zeta \, \frac{1}{(\zeta - z_0)^{n+1}} f(\zeta) \,.$$

(Hint: Theorem 15.8.)

## Exercise 5

Suppose $\mathcal{B}$ is a Banach space and $\mu$ is a bounded positive Borel measure on a compact Hausdorff space $Q$. Consider a sequence $f_n : Q \to \mathcal{B}$, $n \in \mathbb{N}_+$, of continuous functions which satisfy

$$\sum_{n=1}^{\infty} \int_Q \mu(\mathrm{d}x) \, \|f_n(x)\| < \infty \,.$$

Conclude that $g(x) := \sum_{n=1}^{\infty} f_n(x)$ is absolutely summable for almost every $x \in Q$ (with respect to $\mu$), and define $g(x) = 0$ for all other $x \in Q$. Show that $g$ is weakly integrable and that in norm

$$\sum_{n=1}^{N} \int_Q \mu(\mathrm{d}x) \, f_n(x) \overset{N\to\infty}{\longrightarrow} \int_Q \mu(\mathrm{d}x) \, g(x) = \int_Q \mu(\mathrm{d}x) \sum_{n=1}^{\infty} f_n(x) \,,$$

where all integrals are understood in the sense of vector valued integrals. (Hint: Dominated convergence. Note that you cannot assume that $g$ is continuous, so part of the exercise is to prove the existence of $\int_Q \mathrm{d}\mu \, g$.)