Thus if $y = \sum_{\ell=1}^{N} \beta_\ell u_\ell \overset{\Phi^{-1}\text{ lin.}}{\Longrightarrow} \Phi^{-1}(y) = \sum_{\ell=1}^{N} \beta_\ell \Phi^{-1}(\Phi(\hat{e}_\ell))$

$= \sum_{\ell=1}^{N} \beta_\ell \hat{e}_\ell \Rightarrow \gamma_k(y) = \hat{e}_k \cdot \Phi^{-1}(y) = \beta_k \; \forall k.$

Therefore, for any $k = 1, 2, \dots, N$, $\gamma_k : Y \to \mathbb{K}$ is linear and $\ker \gamma_k = \operatorname{span}_{\ell \neq k}(u_\ell) \subsetneq Y$. Since then

$\dim(\ker \gamma_k) = N-1$, we can apply the induction assumption (b) to it. $\Rightarrow \ker \gamma_k$ is closed in Y $\overset{5.3.}{\Rightarrow}$ γ_k is continuous. As $\gamma_k = \operatorname{proj}_k(\Phi^{-1})$ and

$\mathbb{K}^N$ is endowed with the product topology $\Rightarrow \Phi^{-1}$ is continuous. Thus Φ is homeo, which completes the induction step. $\square$

6.3. Theorem: Every locally compact topol. vect. space V has finite dimension.

Proof: Now $\exists V_0 \in \mathcal{T}_V$ s.t. $0 \in V_0$ and $\overline{V_0}$ is compact. By 4.14. b) $\Rightarrow \overline{V_0}$ is bounded $\Rightarrow V_0$ is bounded (just simply because $V_0 \subset \overline{V_0}$.) Therefore, by 4.14. c) the collection $\mathcal{B} := \{2^{-n} V_0 \mid n \in \mathbb{N}_+\}$ is a local base for V. On the other hand, $\{x + \frac{1}{2}V_0\}_{x \in V}$ is an open cover of $\overline{V_0} \Rightarrow \exists N \in \mathbb{N}_+$, $x_k \in V$, $k = 1, 2, \dots, N$, s.t. $\overline{V_0} \subset \bigcup_{k=1}^{N} (x_k + \frac{1}{2}V_0)$.

Let $Y = \operatorname{span}(x_k) \subset V$, which is a subspace of V with $\dim Y \leq N$. If $N = 0 \Rightarrow Y = \{0\}$ = closed. Else, can apply Thm 6.1. Thus we always have that Y is closed. Also, $V_0 \subset \bigcup_{k=1}^{N} (x_k + \frac{1}{2}V_0) \subset Y + \frac{1}{2}V_0$. This proves that $V \subset Y + 2^{-n}V_0$ for $n = 1$. We prove by induction that it holds for all n:

$2^{-n} V_0 \subset 2^{-n}(Y + \frac{1}{2}V_0) = 2^{-n}Y + 2^{-(n+1)}V_0$

$\overset{Y\text{ subspace}}{\subseteq} Y + 2^{-(n+1)}V_0 \Rightarrow V_0 \underset{\text{induction assumpt.}}{\subseteq} Y + 2^{-n}V_0 \subset Y + Y + 2^{-(n+1)}V_0$

$\Rightarrow V_0 \subset Y + 2^{-(n+1)}V_0$, since Y is subspace.

Therefore, we have proven that $V_0 \subset \bigcap_{n=1}^{\infty} (Y + 2^{-n}V_0)$.

$$\Rightarrow V_0 \subset \bigcap_{V\in\mathcal{B}} (Y+V) \overset{\mathcal{B} \text{ is local base}}{=} \bigcap_{\substack{V \text{ open,}\\ 0\in V}} (Y+V) \overset{4.11.a)}{=} \overline{Y} = Y.$$

$$\Rightarrow V_0 \in \mathcal{T}_Y,\ 0\in V_0. \underset{4.14.a)}{\Rightarrow} Y = \bigcup_{n=1}^{\infty} (nV_0) = \mathcal{V}.$$

$$\Rightarrow \dim \mathcal{V} = \dim Y \le N < \infty \quad \square$$

6.4. Corollary: If $\mathcal{V}$ is a locally bounded topol. vect. space with the Heine-Borel property, then $\mathcal{V}$ has finite dimension.

Proof. By assumpt. $\Rightarrow \exists V_0 \in \mathcal{T}_\mathcal{V}$, s.t. $0\in V_0$ and V_0 bounded.
$\underset{4.11.f)}{\Rightarrow} \overline{V_0}$ is bounded. $\underset{\text{Heine-Borel}}{\Rightarrow} \overline{V_0}$ is compact.

Thus $\mathcal{V}$ is actually locally compact. $\Rightarrow \dim \mathcal{V} < \infty$ $\square$

* We have now proven items 4 & 5 in 4.6. (Item 1 was proven in 4.14.c)

7. On invariant metrics

If $\mathcal{V}$ is a metrizable topol. vect. space $\Rightarrow \exists$ metric $d: \mathcal{V}\times\mathcal{V} \to \mathbb{R}$ s.t. $\mathcal{T}_\mathcal{V}$ is induced by d
$\Rightarrow \mathcal{B} := \{B(0,\frac{1}{n})\}_{n\in\mathbb{N}_+}$ form a countable local base at 0. As the following Theorem shows, the existence of a countable local base is also sufficient for a topol. vect. space to be metrizable. (This proves item 2 in 4.6.)

7.1. Theorem: If $\mathcal{V}$ is a topol. vect. space with a countable local base, then there is a metric d on $\mathcal{V}$ such that

(a) d induces $\mathcal{T}_\mathcal{V}$,
(b) $B(0,r)$ is balanced for all $r>0$,
(c) d is invariant.

If, in addition, $\mathcal{V}$ is locally convex, then d can be chosen so that (a), (b), (c) are true, and also

(d) $B(x,r)$ is convex $\forall x\in\mathcal{V}$, $r>0$.

Proof. Rudin, FA, Theorem 1.24. □

Corollary to 7.1.: If $\mathcal{V}$ is a metrizable topol. vect. space $\exists$ metric d as above.

7.2. Theorem If d_1 and d_2 are invariant metrics on a vector space $\mathfrak{X}$, such that the induced topologies are equal (i.e., $\mathcal{T}_{d_1} = \mathcal{T}_{d_2}$ as sets), then

(a) d_1 and d_2 have the same Cauchy sequences

(b) d_1 is complete $\Leftrightarrow$ d_2 is complete.

Proof: Exercise 5.2. □

* The Theorem cannot be applied to metrics which are not translation invariant. (Exercise 5.2.)

7.3. Theorem Consider a subspace $\mathcal{Y}$ of a topol. vect. space $\mathcal{V}$. If $\mathcal{Y}$ is an F-space (in the inherited topology), then $\mathcal{Y}$ is closed in $\mathcal{V}$.

Proof: By assumption, $\exists$ metric d on $\mathcal{Y}$ which is invariant, induces $\mathcal{T}_{\mathcal{Y}}$, and for which every Cauchy sequence in $\mathcal{Y}$ converges. Let $B_\varepsilon := \{ y \in \mathcal{Y} \mid d(y,0) < \varepsilon \}$. Now $\forall n \in \mathbb{N}_+$ $\exists U_n \in \mathcal{T}_{\mathcal{V}}$ s.t. $B_{\frac{1}{n}} = U_n \cap \mathcal{Y} \Rightarrow 0 \in U_n$. By Lemma 4.10. $\exists V_n \in \mathcal{T}_{\mathcal{V}}$ s.t. $0 \in V_n$, $-V_n = V_n$, and $V_n + V_n \subset U_n$.

Consider $x_0 \in \overline{\mathcal{Y}}$, and an arbitrary $W \in \mathcal{T}_{\mathcal{V}}$ with $0 \in W$. Define $E_n(W) := \mathcal{Y} \cap (x_0 + W \cap V_n)$. Since $0 \in W \cap V_n \in \mathcal{T}_{\mathcal{V}}$ and $x_0 \in \overline{\mathcal{Y}}$, every $E_n \neq \emptyset$. But so is every $\bigcap_{k=1}^{n} E_k = \mathcal{Y} \cap (x_0 + W \cap V_1 \cap \dots \cap V_n) \neq \emptyset$, and thus we can choose a sequence (y_n) in $\mathcal{Y}$ s.t. $y_n \in \bigcap_{k=1}^{n} E_k$ $\forall n \in \mathbb{N}_+$. Then, if $m \geq 0 \Rightarrow y_n, y_{n+m} \in E_n \Rightarrow y_n - y_{n+m} = y_n - x_0 - (y_{n+m} - x_0) \in V_n - V_n = V_n + V_n \subset U_n \Rightarrow y_n - y_{n+m} \in U_n \cap \mathcal{Y} = B_{1/n}$ $\Rightarrow d(y_n, y_{n+m}) < \frac{1}{n}$. This shows that (y_n) is Cauchy in $\mathcal{Y} \Rightarrow \exists \bar{y}(W) := \lim_{n\to\infty} y_n \in \mathcal{Y}$. If $\exists n_0$ s.t. $\bar{y} \notin \overline{E}_{n_0}^{(\mathcal{T}_{\mathcal{Y}})}$ $\Rightarrow \exists \varepsilon > 0$ s.t. $\forall y \in E_{n_0}$: $d(\bar{y}, y) \geq \varepsilon$. As $y_n \in E_{n_0}$ for all sufficiently large n, this is a contradiction. $\therefore \bar{y} \in \bigcap_{n=1}^{\infty} \overline{E}_n^{(\mathcal{T}_{\mathcal{Y}})}$.

Suppose $y' \in \bigcap_{n=1}^{\infty} \overline{E}_n^{(\tau_\gamma)}$, $y' \neq \bar{y}$. $\Rightarrow d(y', \bar{y}) > 0 \Rightarrow$
$\exists n$ s.t. $d(y', \bar{y}) > \frac{3}{n}$. Since $y', \bar{y} \in \overline{E}_n^{(\tau_\gamma)}$, $\exists y_1', y_2' \in E_n$
s.t. $d(y', y_1') < \frac{1}{n}$, $d(\bar{y}, y_2') < \frac{1}{n} \Rightarrow d(y', \bar{y}) \leq \frac{2}{n} + d(y_1', y_2')$,
where $d(y_1', y_2') = d(\underbrace{y_1' - y_2'}_{\in V_n + V_n \subset U_n}, 0) < \frac{1}{n} \Rightarrow d(y', \bar{y}) \leq \frac{3}{n}$ ↯

Thus $\bigcap_{n=1}^{\infty} \overline{E}_n^{(\tau_\gamma)} = \{\bar{y}(W)\}$. Denote $\bar{y} := \bar{y}(V)$. As every

$E_n(W) \subset E_n(V) \Rightarrow \bigcap_{n=1}^{\infty} \overline{E}_n^{(\tau_\gamma)}(W) \subset \bigcap_{n=1}^{\infty} \overline{E}_n^{(\tau_\gamma)}(V) = \{\bar{y}\}$.

Thus $\bar{y}(W) = \bar{y}$ $\forall W$. Therefore, $\bar{y} \in \bigcap_{\substack{W \in T_V \\ 0 \in W}} (x_0 + W) = \overline{\{x_0\}}^{(\tau_\gamma)} = \{x_0\}$
$\Rightarrow \bar{y} = x_0 \Rightarrow x_0 \in Y$ $\square$

7.4. Proposition

(a) If d is a translation invariant metric on a vector space X, then

$$(*) \qquad d(nx, 0) \leq n\, d(x, 0) \quad \forall\, x \in X,\ n \in \mathbb{N}_+$$

(b) Consider a sequence (x_n) in V which is a metrizable topol. vect. space. If $x_n \to 0$ as $n \to \infty$, then for all $n \in \mathbb{N}_+$ $\exists \gamma_n > 0$ s.t. $\gamma_n \to \infty$ and $\gamma_n x_n \to 0$.

Proof. (a) Since $d((n+1)x, 0) \leq d((n+1)x, nx) + d(nx, 0)$
$= d(x, 0) + d(nx, 0)$, an easy induction proves $(*)$.

(b) By Corollary to 7.1. there is now an invariant metric d compatible with the topology of V. Then $d(x_n, 0) \xrightarrow{n \to \infty} 0$ and thus $\forall k \in \mathbb{N}_+$ $\exists n_k \in \mathbb{N}$ s.t. $d(x_n, 0) \leq \frac{1}{k^2}$ $\forall n \geq n_k$.
Define $m_k := \max_{\ell = 1, \ldots, k} \{n_\ell, k\}$ and set $\gamma_n = 1$, for $n < m_1$,

$\gamma_n = k$, if $m_k \leq n < m_{k+1}$. (For any $n \geq m_1$, there exists a unique k satisfying $m_k \leq n < m_{k+1}$, since (m_k) is an increasing sequence in $\mathbb{N}_+$ and $m_k \xrightarrow{k \to \infty} \infty$.)
Then clearly $\gamma_n \xrightarrow{n \to \infty} \infty$, and for $n \geq m_1$:

$$d(\gamma_n x_n, 0) = d(k x_n, 0) \underset{a)}{\leq} k\, d(x_n, 0) \underset{n \geq m_k \geq n_k}{\leq} k \cdot \frac{1}{k^2} = \frac{1}{k} = \frac{1}{\gamma_n} \xrightarrow{n \to \infty} 0 \quad \square$$

8. Boundedness and continuity of linear maps

8.1. Proposition (Properties of topologically bounded sets)

Let $\mathcal{V}$ be a topol. vect. space, and $E \subset \mathcal{V}$.

(a) E compact $\Rightarrow$ E bounded and closed
(b) E bounded $\Rightarrow$ $\overline{E}$ bounded
(c) If $\exists B \subset \mathcal{V}$ s.t. B is bounded and $E \subset B$
$\Rightarrow$ E bounded
(d) E is a bounded subspace
$\Leftrightarrow$ $E = \{0\}$
(e) If $E = \{x_n\}$, where $(x_n)_{n \in \mathbb{N}_+}$ is a Cauchy sequence, then E is bounded. If (x_n) is convergent, then E is bounded.
(f) E bounded
$\Leftrightarrow$ [If $x_n \in E$, $\alpha_n \in \mathbb{K}$, $n \in \mathbb{N}_+$, and $\alpha_n \xrightarrow{n\to\infty} 0$, then $\alpha_n x_n \xrightarrow[n\to\infty]{} 0$.]

Proof. (a) As $\mathcal{V}$ is Hausdorff, E is closed. Boundedness was proved in 4.11.b).
(b) was proved in 4.11.f).
(c) follows directly from the definition. ∎
(d) $E = \{0\}$ is a subspace, and if $t > 0$, $V \in T_{\mathcal{V}}$, $0 \in V$, then $0 = t0 \in tV \Rightarrow E \subset tV$.
Thus E is bounded.
If E is a subspace and $x \in E$, $x \neq 0$, then $\exists V \in T_{\mathcal{V}}$ s.t. $0 \in V$ and $x \notin V$ (Hausdorff.)
$\Rightarrow nx \notin nV$ $\forall n \in \mathbb{N}_+$. Since $nx \in E$ $\forall n$, this shows that E is not bounded. ∎
(e) Suppose $(x_n)_{n \in \mathbb{N}_+}$ is Cauchy $\Leftrightarrow$ $\forall V \in T_{\mathcal{V}}$ with $0 \in V$, $\exists n_V \in \mathbb{N}_+$ s.t. $x_n - x_m \in V$ $\forall n, m \geq n_V$.
Now, if $V_0 \in T_{\mathcal{V}}$, $0 \in V_0$, then $\exists U, W \in T_{\mathcal{V}}$ s.t. $0 \in U$, $0 \in W$, both are balanced and $W + W \subset U \subset V_0$. Then $x_n \in x_{n_W} + W$ $\forall n \geq n_W$.
By Prop. 4.14.a) $E \subset \bigcup_{n=1}^{\infty} (nW)$. Thus $\exists M \geq 1$ s.t. $x_n \in \bigcup_{n=1}^{M} (nW)$ for all $1 \leq n \leq n_W$. Since W is balanced $\Rightarrow$ $\bigcup_{n=1}^{M} (nW) = MW$. Now if $n > n_W$: $x_n \in x_{n_W} + W \in MW + W \underset{M\geq 1}{\subset} MW + MW \subset M(W+W) \subset MU$.

(*) $x_n \to x \Rightarrow x_n - x_m = x_n - x - (x_m - x) \in W - W \Rightarrow (x_n)$ is Cauchy.

As $0 \in W$, $W \subset W + W \subset U$, and thus we have proven that $E \subset MU$. Now $\forall t \geq M$ we have
$E \subset \frac{M}{t} tU \underset{\uparrow U \text{ bal.}}{\subset} tU \subset tV_0$, i.e. E bounded. ∎

?(*)

(f) If E bounded and $V \in \mathcal{T}_V$, balanced and $0 \in V$, then $\exists t > 0$ s.t. $E \subset tV$. Now for $x_n \in E$, $\alpha_n \to 0$, there is $N \in \mathbb{N}_+$ s.t. $\forall n \geq N$: $|\alpha_n| t < 1$. Thus $\forall n \geq N$:
$\alpha_n x_n = \alpha_n t \frac{1}{t} x_n \in \underbrace{\alpha_n t}_{|\,| < 1} V \underset{\uparrow V \text{ bal.}}{\subseteq} V$. Since any nborhood of 0 contains a balanced nborhood of 0, this implies that $\alpha_n x_n \underset{n \to \infty}{\to} 0$.

If E is not bounded, $\exists V \in \mathcal{T}_V$ s.t. $0 \in V$ and $\forall n \in \mathbb{N}_+ \; \exists r_n > n$ s.t. $E \cap (r_n V)^c \neq \emptyset$. Choose $x_n \in E \cap (r_n V)^c$ and let $\alpha_n = r_n^{-1}$. Then $\{x_n\} \subset E$, $\alpha_n \overset{n \to \infty}{\to} 0$ but $\alpha_n x_n = r_n^{-1} x_n \notin r_n^{-1} r_n V = V$, and thus $\alpha_n x_n \not\to 0$ □

8.2. Definition: A linear map $\Lambda : V_1 \to V_2$ between topol. vector spaces V_1, V_2 is called bounded if $\Lambda(E)$ is bounded $\forall E \subset V_1$ bounded.

8.3. Theorem Let V_1 and V_2 be topol. vect. spaces and $\Lambda : V_1 \to V_2$ linear. Then
(a) ⇒ (b) ⇒ (c) and (a) ⇒ (d), where

(a) Λ is continuous
(b) Λ is bounded
(c) If $x_n \to 0$ then $\{\Lambda x_n\}_{n \in \mathbb{N}_+}$ is bounded.
(d) If $x_n \to 0$ then $\Lambda x_n \to 0$.

In case V_1 is metrizable, then all four conditions are equivalent.

Proof: (a) ⇒ (d) is obvious.
"(a) ⇒ (b)" Suppose $E \subset V_1$ is bounded, Λ contin.
Consider $W \in \mathcal{T}_{V_2}$, $0 \in W$. As $\Lambda 0 = 0$ ⇒

$\exists V \in \mathcal{T}_{\mathcal{V}_1}$ s.t. $0 \in V$ and $\Lambda(V) \subset W$. As E is bounded, $\exists t_0 > 0$ s.t. $E \subset tV$ $\forall t > t_0$. Thus $\Lambda(E) \subset \Lambda(tV) \subset t\Lambda(V) \subset tW$ $\forall t > t_0$. Thus $\Lambda(E)$ is bounded. ■

"b) ⇒ c)" $x_n \to 0 \underset{8.1.e)}{\Rightarrow} \{x_n\}_{n\in\mathbb{N}}$ is bounded $\underset{\text{assumption}}{\Rightarrow} \{\Lambda x_n\}$ bnded. ■

Thus the first implications have been proven. For the remaining, assume $\mathcal{V}_1$ is metrizable ⇒ ∃ invariant metric d_1 on $\mathcal{V}_1$ which induces $\mathcal{T}_{\mathcal{V}_1}$.

"c) ⇒ d)" Suppose $x_n \to 0 \overset{7.4b)}{\Rightarrow} \exists \gamma_n > 0$ s.t. $\gamma_n \to \infty$ and $\gamma_n x_n \to 0 \underset{c)}{\Rightarrow} \{\Lambda(\gamma_n x_n)\}$ is bounded.

As $\gamma_n^{-1} \to 0$, by 8.1.f) ⇒ $\Lambda(x_n) = \gamma_n^{-1}\Lambda(\gamma_n x_n) \to 0$ ■

"d) ⇒ a)" Suppose a) fails. $\overset{5.3.}{\Rightarrow}$ Λ is not contin. at 0. Thus $\exists W \in \mathcal{T}_{\mathcal{V}_2}$ s.t. $0 \in W$ but $U \not\subset \Lambda^{\leftarrow}W$ $\forall U \in \mathcal{T}_{\mathcal{V}_1}$ with $0 \in U$. ⇒ $\forall n \in \mathbb{N}$ $\exists x_n \in B(0, \frac{1}{n})$ s.t. $x_n \notin \Lambda^{\leftarrow}W$. But then $d_1(x_n, 0) < \frac{1}{n}$ ⇒ $x_n \to 0$, but $\Lambda x_n \notin W$ and thus $\Lambda x_n \not\to 0$. This contradicts d). □

The following extension theorems are also useful in many applications.

<u>8.4. Theorem</u> Let $\mathcal{V}$ and $\mathcal{F}$ be topol. vect. spaces, and assume that $\mathcal{F}$ is a F-space. Suppose $M \subset \mathcal{V}$ is a <u>dense</u> subspace, and $\Lambda : M \to \mathcal{F}$ is continuous and linear. Then there is a unique $\bar{\Lambda} : \mathcal{V} \to \mathcal{F}$ which is continuous and $\bar{\Lambda}|_M = \Lambda$. In addition, $\bar{\Lambda}$ is linear.

Proof. Exercise 5.3. □

<u>Corollary</u>: If A is a bounded, densely defined operator on a Hilbert space $\mathcal{H}$, then it has a unique extension $\bar{A} \in \mathcal{B}(\mathcal{H})$. In addition, $\|\bar{A}\| = \|A\|$.

<u>Proof.</u> Exercise 5.3. □