b) By 4.12 a) $\exists W \subset V_0$ s.t. W is balanced, open and $0 \in W$. By a), $K \subset V = \bigcup_{n=1}^{\infty} (nW)$.

As K is compact, $\exists n_0 \in \mathbb{N}_+$ s.t.
$K \subset \bigcup_{n=1}^{n_0} (nW) = n_0 W$. ($n \le n_0$, $x \in nW \Rightarrow \exists x_0 \in W$ s.t. $x = n x_0 = n_0 \frac{n}{n_0} x_0 \underset{\uparrow \, W \text{ balanced.}}{\in} n_0 W$.)

Therefore, if $t > n_0$ and $x \in K \Rightarrow x \in n_0 W = t \frac{n_0}{t} W \subset tW$, as W is balanced.
$\therefore$ $K \subset tV_0$. Since V_0 was an arbitrary neighb. of 0, this proves that K is bounded. ■

c) Consider $U \in \mathcal{T}_V$ s.t. $0 \in U$. If V_0 is bounded, $\exists s > 0$ s.t. $V_0 \subset tU$ $\forall t > s$. Then $\exists n_0 \in \mathbb{N}_+$ s.t. $s\delta_n < 1$ $\forall n \ge n_0$. ($s\delta_n \to 0^+$.) $\Rightarrow V_0 \subset \delta_n^{-1} U$ $\forall n \ge n_0$.
$\Rightarrow \delta_{n_0} V_0 \subset U$. Since each $\delta_n V_0$ is open and contains 0, this proves that $\mathcal{B}$ is a local base. □

5. Linear maps between topol. vect. spaces

* Reminders: For $f : X \to Y$, $A \subset X$, $B \subset Y$
  a) image of $A = f(A) := \{ f(x) \mid x \in A \}$
  b) preimage of $B = f^{\leftarrow}(B) := \{ x \in X \mid f(x) \in B \}$
  c) In case X, Y are vector spaces (both over $\mathbb{K}$)
  f is linear $\Leftrightarrow f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ $\forall \alpha, \beta \in \mathbb{K}$, $x, y \in X$.

  Shorthand notation: $fx := f(x)$
  (used when f is linear.)

* Definition: If X is a vector space over $\mathbb{K}$, any linear $f : X \to \mathbb{K}$ is called a <u>linear functional</u>.

* For linear functional $\Lambda : X \to \mathbb{K}$, $\Lambda = 0$ refers to the map $\Lambda x = 0 \in \mathbb{K}$ $\forall x \in X$. (which is obviously linear).

5.1. Proposition Consider vector spaces $\mathfrak{X}, \mathfrak{Y}$, and subsets $A \subset \mathfrak{X}$, $B \subset \mathfrak{Y}$.
If $\Lambda : \mathfrak{X} \to \mathfrak{Y}$ is linear, then

(a) $\Lambda 0_{\mathfrak{X}} = 0_{\mathfrak{Y}}$
(b) A subspace $\Rightarrow \Lambda(A)$ subspace
A convex $\Rightarrow \Lambda(A)$ convex
A balanced $\Rightarrow \Lambda(A)$ balanced
(c) B subspace $\Rightarrow \Lambda^{\leftarrow}(B)$ subspace
B convex $\Rightarrow \Lambda^{\leftarrow}(B)$ convex
B balanced $\Rightarrow \Lambda^{\leftarrow}(B)$ balanced

Proof. Obvious computations. For instance, if B is balanced, $|\alpha| \le 1$, and $y \in \Lambda^{\leftarrow}(B)$
$\Rightarrow \Lambda(\alpha y) = \alpha \Lambda y \in \alpha B \underset{\uparrow \text{B balanced}}{\subset} B \Rightarrow \alpha y \in \Lambda^{\leftarrow}(B)$.
$\therefore \alpha \Lambda^{\leftarrow}(B) \subset \Lambda^{\leftarrow}(B)$. $\square$

Corollaries: a) $\operatorname{Ker} \Lambda := \Lambda^{\leftarrow}(\{0\})$ is a subspace.
b) $R(\Lambda) := \Lambda(\mathfrak{X})$ is a subspace.

5.2. Theorem Let $\mathcal{V}_1, \mathcal{V}_2$ be topol. vect. spaces. If $\Lambda : \mathcal{V}_1 \to \mathcal{V}_2$ is linear and continuous at $0_{\mathcal{V}_1}$, then Λ is continuous.
In addition, then Λ is even uniformly continuous: For every $W \in \mathcal{T}_{\mathcal{V}_2}$ with $0 \in W$, $\exists V \in \mathcal{T}_{\mathcal{V}_1}$ with $0 \in V$

s.t. $[\, y - x \in V$ implies $\Lambda y - \Lambda x \in W\,]$.

Proof. Consider $W \in \mathcal{T}_{\mathcal{V}_2}$ with $0_{\mathcal{V}_2} \in W$. Since $\Lambda 0_{\mathcal{V}_1} = 0_{\mathcal{V}_2}$, by the assumed continuity $\exists V \in \mathcal{T}_{\mathcal{V}_1}$ s.t. $0_{\mathcal{V}_1} \in V$ and $V \subset \Lambda^{\leftarrow} W$. Now if $y, x \in \mathcal{V}_1$ are such that $y - x \in V$, we have $\Lambda y - \Lambda x = \Lambda(y-x) \in \Lambda V \subset W$. This proves the 2nd statement. To prove continuity, consider $U \in \mathcal{T}_{\mathcal{V}_2}$, $x \in \Lambda^{\leftarrow} U$. Set $W = U - \Lambda x$. $\Rightarrow 0 \in W$ and $W \in \mathcal{T}_{\mathcal{V}_2}$ $\Rightarrow \exists V$ as above
$\Rightarrow \Lambda(x + V) = \Lambda x + \Lambda V \subset \Lambda x + W = U$
$\Rightarrow x \in x + V$, $x + V$ is open, and $x + V \subset \Lambda^{\leftarrow} U$.
$\therefore \Lambda$ is continuous. $\square$

5.3. Theorem Let $\Lambda: V \to \mathbb{K}$ be a linear functional on a topol. vector space V. If $\Lambda \neq 0$, the following statements are equivalent:

(a) Λ is continuous.
(b) $\ker \Lambda$ is closed.
(c) $\ker \Lambda$ is not dense in V.
(d) Λ is bounded in some neighborhood W of 0_V.
$\Leftrightarrow \sup_{x \in W} |\Lambda x| < \infty$.

Proof: "(a) ⇒ (b)": If Λ contin. $\Rightarrow \ker \Lambda = \Lambda^{\leftarrow}(\{0\})$ is closed.

"(b) ⇒ (c)": If $\ker \Lambda = \overline{\ker \Lambda} = V \Rightarrow \Lambda = 0$ which was excluded in the assumptions. Thus $\neg(c) \Rightarrow \neg(b)$.

"(c) ⇒ (d)": Assume (c) holds. $\Rightarrow (\overline{\ker \Lambda})^c$ is open and non-empty. $\Rightarrow \exists x_0 \in V$ and $W \in \mathcal{T}_V$ s.t. $0 \in W$ and $(x_0 + W) \cap \overline{\ker \Lambda} = \emptyset$
$\Rightarrow (x_0 + W) \cap \ker \Lambda = \emptyset$. By 4.12. a) we can choose W to be balanced.
$\overset{5.1.b)}{\Rightarrow} \Lambda(W)$ is balanced. Suppose $\Lambda(W)$ is not bounded. Then $\forall x \in \mathbb{K}$, there is $y \in \Lambda(W)$ s.t. $|y| > |x|$, and thus $x = \frac{x}{y} y \in \underbrace{\frac{x}{y}}_{\in \mathbb{K}\ \&\ |\frac{x}{y}|<1} \Lambda(W) \subset \Lambda(W)$.

Therefore, then $\Lambda(W) = \mathbb{K} \Rightarrow \exists y \in W$ s.t. $\Lambda y = -\Lambda x_0 \Rightarrow x_0 + y \in \ker \Lambda$. But this is a contradiction $\Rightarrow \Lambda(W)$ is bounded.

"(d) ⇒ (a)": If (d) holds, $\exists W \in \mathcal{T}_V$ s.t. $0_V \in W$ and $\Lambda(W)$ is bounded. $\Rightarrow \exists M > 0$ s.t. $|\Lambda x| < M \ \forall x \in W$. Now $\Lambda 0_V = 0$ and if $U \subset \mathbb{K}$ open and $0 \in U \Rightarrow \exists \varepsilon > 0$ s.t. $B(0, \varepsilon) \subset U$. But then for $V = \frac{\varepsilon}{M} W$ we have $0_V \in V$, V open, and $x \in V$
$\Rightarrow |\Lambda x| < \frac{\varepsilon}{M} \cdot M = \varepsilon$, implying $V \subset \Lambda^{\leftarrow} U$.
This proves that Λ is contin. at 0
$\overset{5.2.}{\Rightarrow} \Lambda$ is continuous. □

6. Finite-dimensional spaces

6.1. Theorem Suppose V is a topol. vect. space, $Y \subset V$ is a subspace, and $N := \dim Y$ satisfies $0 < N < \infty$. Then

a) every vector-space-isomorphism $\Phi : \mathbb{K}^N \to Y$ is a homeomorphism

b) Y is closed.

* Preliminaries: Topology reminder

* If $Y \subset X$ = space with topology $\mathcal{T}_X$, its relative or inherited topology is $\mathcal{T}_Y := \{ U \cap Y \mid U \in \mathcal{T}_X \}$.

* If $f : X \to X'$ is continuous and $Y \subset X$, $Y' \subset X'$, $f(Y) \subset Y'$, then $f|_Y : Y \to X'$ and $f_1 : Y \to Y'$ are continuous. ($f_1(x) = f(x)\ \forall x \in Y$)

6.2. Lemma: Suppose V is a topol. vect. space and $Y \subset V$ is a subspace. Then Y is a topol. vect. space. If Y is locally compact, then it is closed in V.

Proof: The first stament follows from the above reminder.

By assumption, $\exists V_0 \in \mathcal{T}_Y$ s.t. $0 \in V_0$ and $K := \overline{V_0}^{(\mathcal{T}_Y)}$ is compact. Now $\exists U_0 \in \mathcal{T}_V$ s.t. $V_0 = U_0 \cap Y \subset K$. Here $0 \in U_0$. By Corollary 4.9., there is $W' \in \mathcal{T}_V$ s.t. $0 \in W'$ and $\overline{W'} \subset U_0$. By continuity of addition, $\exists V_1, V_2 \in \mathcal{T}_V$ s.t. $0 \in V_1 \cap V_2$ and $V_1 + V_2 \subset W'$. As $V_1 \cap V_2$ is also open, there is $W_0 \in \mathcal{T}_V$ s.t. $0 \in W_0$, W_0 is balanced and $W_0 \subset V_1 \cap V_2 \Rightarrow W_0 + W_0 \subset W'$.

$\Rightarrow \overline{W_0} + \overline{W_0} \subset \overline{W'} \subset U_0$. (By 4.11. e) also $\overline{W_0}$ is balanced.)

$\uparrow$ 4.11. b)

Let us prove next that $Y \cap (x + \overline{W_0}) =: K_x$ is compact $\forall x \in V$. Consider $y_0, y \in K_x$.

$\Rightarrow y - y_0 = (y - x) + [-(y_0 - x)] \in \overline{W_0} + \overline{W_0} \subset U_0$.

Since Y is a subspace, also $y - y_0 \in Y$, and thus $y - y_0 \in U_0 \cap Y = V_0 \subset K$. This implies $K_x \subset y_0 + K$, where $y_0 + K$ is a compact set. However, since $x + \overline{W_0}$ is closed in $\mathcal{T}_V$, K_x is closed in $\mathcal{T}_Y$, and thus K_x is compact. ($K_x = \emptyset$ is also ok.)

Next fix $x \in \bar{Y}$. Define $\mathcal{B} := \{V \in \mathcal{T}_V \mid 0 \in V, V \subset W_0\}$ and $E_V := Y \cap (x + \bar{V})$ $\forall V \in \mathcal{B}$. Then $E_V \subset K_x$, and as above, we can conclude that E_V is compact. If $E_V = \emptyset \Rightarrow Y \cap \underbrace{(x+V)}_{\text{open}} = \emptyset \Rightarrow \bar{Y} \cap (x+V) = \emptyset$ which
↑ see p. 48

is not true as x belongs to both $\bar{Y}$ and $x+V$. Thus every $E_V \neq \emptyset$. Now if $V_i \in \mathcal{B}$, for $i = 1, 2, \ldots, n$, $n < \infty$, then $V := \bigcap_{i=1}^n V_i$ is open, contains 0 and $\subset W_0 \Rightarrow V \in \mathcal{B}$ and thus E_V is defined. Also $y \in \bigcap_{i=1}^n E_{V_i} \Leftrightarrow y \in Y$ and $y - x \in \bar{V}_i$ $\forall i$.

It is straightforward to check that $\bar{V} = \bigcap_{i=1}^n \bar{V}_i$:

$V \subset \bigcap_{i=1}^n \bar{V}_i$ = closed $\Rightarrow \bar{V} \subset \bigcap \bar{V}_i$. If $x' \in \bigcap_{i=1}^n \bar{V}_i$

and $W \in \mathcal{T}_V$, $0 \in W$ arbitrary $\Rightarrow (x' - W) \cap V_i \neq \emptyset$ $\forall i$
$\Rightarrow (x' - W) \cap V \neq \emptyset \Rightarrow \exists y' \in W$, $v \in V$ s.t. $v = x' - y'$
$\Rightarrow x' = v + y' \in V + W$. By 4.11.a) $\Rightarrow x' \in \bar{V}$.

Therefore, $\bigcap_{i=1}^n E_{V_i} = \{y \in Y \mid y - x \in \bar{V}\} = E_V$. This shows that $\{E_V \mid V \in \mathcal{B}\}$ is a collection of compact, non-empty sets with "finite intersection property". We shall prove in (*) that this implies that $\bigcap_{V \in \mathcal{B}} E_V \neq \emptyset$.

If $z \in \bigcap_{V \in \mathcal{B}} E_V \Rightarrow z \in Y$ and $z \in x + \bar{V}$ $\forall V \in \mathcal{B}$.

If $z \neq x$, by Corollaries 4.8. & 4.9. there is $U' \in \mathcal{T}_V$ s.t. $0 \in U'$ and $(x + \bar{U'}) \cap (z + \bar{U'}) = \emptyset$. But then for $V' := U' \cap W$ we have $0 \in V'$, $V' \in \mathcal{T}_V$, $V' \subset W$ $\Rightarrow V' \in \mathcal{B}$, and $z \in (x + \bar{U'})^c \subset (x + \bar{V'})^c$. This is a contradiction, and thus $z = x \Rightarrow x \in Y$.
$\therefore$ Y is closed. $\square$

(*): Clearly, $W_0 \in \mathcal{B} \Rightarrow E_{W_0}$ is compact. Suppose $\bigcap_{V \in \mathcal{B}} E_V = \emptyset$.
$\Rightarrow E_{W_0} \subset Y = \bigcup_{V \in \mathcal{B}} E_V^c$. As each E_V is closed in Y, we can choose a finite subcover $\Rightarrow \exists V_i \in \mathcal{B}$, $i = 1, \ldots, n$,
s.t. $E_{W_0} \subset \bigcup_{i=1}^n E_{V_i}^c = (\bigcap_{i=1}^n E_{V_i})^c = E_{\cap V_i}^c \Rightarrow E_{W_0} \cap E_{\cap V_i} = E_{W_0 \cap (\bigcap_{i=1}^n V_i)} = \emptyset$. $\square$ contradiction.

Proof of Theorem 6.1.

① In general, a) $\Rightarrow$ b) by Lemma 6.2.:
Suppose a) holds. Consider $B_1 := B(0,1) \subset \mathbb{K}^N$
($B(x,\varepsilon)$ = Ball centered at x with radius ε
$= \{y \mid d(x,y) < \varepsilon\}$.)
Then $0 \in B_1$, B_1 is open and $\overline{B}_1$ is compact in $\mathbb{K}^N$. This proves that $\mathbb{K}^N$ is locally compact.
Let $U := \Phi(B_1) \subset Y$. As Φ is homeo, U is open and $\overline{U} = \overline{\Phi(B_1)} = \Phi(\overline{B}_1)$ is compact.
Also, by linearity of Φ, $\Phi(0) = 0 \Rightarrow 0 \in U$.
Thus Y is locally compact. $\underset{6.2}{\Rightarrow}$ Y is closed in V.

② Assume $N=1$, and let $\Phi : \mathbb{K} \to Y$ be linear and bijective $\Rightarrow \Phi^{-1}$ is linear. Define $u_1 := \Phi(1) \in Y$, when by linearity $\Phi(\alpha) = \alpha u_1 \ \forall \alpha \in \mathbb{K}$.
By continuity of scalar multiplication, Φ is continuous. Since Φ^{-1} is linear bijection, $\operatorname{Ker} \Phi^{-1} = \{0\}$ = closed set in Y. By 5.3.
$\Rightarrow \Phi^{-1}$ is continuous. $\therefore \Phi$ is homeom.
$\underset{①}{\Rightarrow}$ Y is closed. ∎

③ Make the induction assumption that a) & b) hold for subspaces with $\dim \le N-1$, $N > 1$. It suffices to prove that a) then holds for $\dim = N$, since by ① this implies that b) holds for $\dim = N$, as well.

Consider thus $\Phi : \mathbb{K}^N \to Y$ which is linear and bijective. Define $u_k := \Phi(\hat{e}_k)$, $k = 1, 2, \dots, N$, when by linearity of Φ

$$\Phi(x) = \sum_{k=1}^{N} \alpha_k(x) u_k, \quad \text{where } \alpha_k(x) := \hat{e}_k \cdot x.$$
$$\left(\Rightarrow x = \sum_{k=1}^{N} \alpha_k(x) \hat{e}_k. \right)$$

Since each $\alpha_k : \mathbb{K}^N \to \mathbb{K}$ is continuous, every $x \mapsto \alpha_k(x) u_k$ is contin. (as in ②) and so is their sum (since addition is continuous). $\Rightarrow \Phi$ is contin.
Every α_k is also linear $\Rightarrow \gamma_k := \alpha_k \circ \Phi^{-1}$ is linear.
Also if $y \in Y \Rightarrow \Phi^{-1}(y) = \sum_{k=1}^{N} \alpha_k(\Phi^{-1} y) \hat{e}_k = \sum_{k=1}^{N} \gamma_k(y) \hat{e}_k$.

Thus if $y = \sum_{\ell=1}^{N} \beta_\ell u_\ell \overset{\Phi^{-1}\text{ lin.}}{\Longrightarrow} \Phi^{-1}(y) = \sum_{\ell=1}^{N} \beta_\ell \Phi^{-1}(\Phi(\hat{e}_\ell))$

$= \sum_{\ell=1}^{N} \beta_\ell \hat{e}_\ell \Rightarrow \gamma_k(y) = \hat{e}_k \cdot \Phi^{-1}(y) = \beta_k \;\forall k.$

Therefore, for any $k = 1, 2, \dots, N$, $\gamma_k : Y \to \mathbb{K}$ is linear and $\ker \gamma_k = \operatorname{span}_{\ell \neq k}(u_\ell) \subsetneq Y$. Since then

$\dim(\ker \gamma_k) = N-1$, we can apply the induction assumption (b) to it. $\Rightarrow$ $\ker \gamma_k$ is closed in Y $\overset{5.3.}{\Rightarrow}$ γ_k is continuous. As $\gamma_k = \mathrm{proj}_k(\Phi^{-1})$ and

$\mathbb{K}^N$ is endowed with the product topology $\Rightarrow$ Φ^{-1} is continuous. Thus Φ is homeo, which completes the induction step. □

6.3. Theorem: Every locally compact topol. vect. space $\mathcal{V}$ has finite dimension.

Proof: Now $\exists V_0 \in \mathcal{T}_\mathcal{V}$ s.t. $0 \in V_0$ and $\overline{V_0}$ is compact. By 4.14. b) $\Rightarrow$ $\overline{V_0}$ is bounded $\Rightarrow$ V_0 is bounded (just simply because $V_0 \subset \overline{V_0}$.) Therefore, by 4.14. c) the collection $\mathcal{B} := \{2^{-n} V_0 \mid n \in \mathbb{N}_+\}$ is a local base for $\mathcal{V}$. On the other hand, $\{x + \frac{1}{2}V_0\}_{x \in \mathcal{V}}$ is an open cover of $\overline{V_0}$ $\Rightarrow$ $\exists N \in \mathbb{N}_+$, $x_k \in \mathcal{V}$, $k = 1, 2, \dots, N$, s.t. $\overline{V_0} \subset \bigcup_{k=1}^{N} (x_k + \frac{1}{2}V_0)$.

Let $Y = \operatorname{span}(x_k) \subset \mathcal{V}$, which is a subspace of $\mathcal{V}$ with $\dim Y \leq N$. If $N = 0$ $\Rightarrow$ $Y = \{0\}$ = closed. Else, can apply Thm 6.1. Thus we always have that Y is closed. Also, $V_0 \subset \bigcup_{k=1}^{N} (x_k + \frac{1}{2}V_0) \subset Y + \frac{1}{2}V_0$. This proves that $V \subset Y + 2^{-n}V_0$ for $n = 1$. We prove by induction that it holds for all n:

$$2^{-n}V_0 \subset 2^{-n}(Y + \tfrac{1}{2}V_0) = 2^{-n}Y + 2^{-(n+1)}V_0$$

$$\underset{Y\text{ subspace}}{\subset} Y + 2^{-(n+1)}V_0 \Rightarrow V_0 \underset{\text{induction assumpt.}}{\subset} Y + 2^{-n}V_0 \subset Y + Y + 2^{-(n+1)}V_0$$

$\Rightarrow V_0 \subset Y + 2^{-(n+1)}V_0$, since Y is subspace.

Therefore, we have proven that $V_0 \subset \bigcap_{n=1}^{\infty} (Y + 2^{-n}V_0)$.

$$\Rightarrow V_0 \subset \bigcap_{V \in \mathcal{B}} (Y+V) \overset{\mathcal{B} \text{ is local base}}{=} \bigcap_{\substack{V \text{ open,} \\ 0 \in V}} (Y+V) \overset{4.11.a)}{=} \overline{Y} = Y.$$

$$\Rightarrow V_0 \in \mathcal{T}_Y,\ 0 \in V_0. \underset{4.14.a)}{\Rightarrow} Y = \bigcup_{n=1}^{\infty} (nV_0) = \mathcal{V}.$$

$$\Rightarrow \dim \mathcal{V} = \dim Y \leq N < \infty \quad \square$$

<u>6.4. Corollary</u> If $\mathcal{V}$ is a locally bounded topol. vect. space with the Heine-Borel property, then $\mathcal{V}$ has finite dimension.

<u>Proof.</u> By assumpt. $\Rightarrow \exists V_0 \in \mathcal{T}_{\mathcal{V}}$ s.t. $0 \in V_0$ and V_0 bounded. $\underset{4.11.f)}{\Rightarrow} \overline{V}_0$ is bounded. $\underset{\text{Heine-Borel}}{\Rightarrow} \overline{V}_0$ is compact.

Thus $\mathcal{V}$ is actually locally compact. $\Rightarrow \dim \mathcal{V} < \infty$ $\square$

* We have now proven item 5 in 4.6.