2.13. Correction

The correct definition of the "internal" direct sum of subspaces of Hilbert-spaces should read:

* $M_1, M_2 \subset \mathcal{H}$ are closed subspaces
* $\psi \perp \phi \quad \forall \psi \in M_1, \phi \in M_2$

This has the benefit that

$$M_1 \oplus_{\text{internal}} M_2 \cong M_1 \oplus_{\text{external}} M_2$$

via the obvious isomorphism $(\psi_1, \psi_2) \in M_1 \times M_2 \longmapsto \psi_1 + \psi_2 \in M_1 + M_2 \subset \mathcal{H}$

(Proof: Bijectivity and linearity are obvious, as a map $M_1 \oplus_{ext} M_2 \to M_1 + M_2$.
For scalar product: $((\psi_1, \psi_2), (\psi_1', \psi_2'))_{M_1 \oplus_{ext} M_2}$
$= (\psi_1, \psi_1')_{\mathcal{H}} + (\psi_2, \psi_2')_{\mathcal{H}}$
$= (\psi_1, \psi_1') + (\psi_2, \psi_2') + (\psi_1, \psi_2') + (\psi_2, \psi_1')$
$= (\psi_1 + \psi_2, \psi_1' + \psi_2')_{\mathcal{H}}$. $\square$)

Remarks:

* The previous definition is used for general vector spaces. The main problem with it is that $M_1 + M_2$ does not need to be closed $\Rightarrow$ it is not a Hilbert space.
(See, e.g., Schochetman, Smith, Tsui: Int. J. Math. Math. Sci. 26 (2001) 257-267)

Also, even if $M_1 + M_2$ is closed, the simple map above would typically not be an isomorphism.

* Note that the statements in 2.13. assume this definition. However, 2.16. is OK for either definition.

* Note that "$M_1 \oplus M_2$"$|_{new}$ $\Rightarrow$ "$M_1 \oplus M_2$"$|_{old}$ or in the vector space sense: As M_1, M_2 are subspaces, $0 \in M_1 \cap M_2$. But $u \in M_1 \cap M_2 \Rightarrow (u,u) = 0 \Rightarrow u = 0$. Thus $M_1 \cap M_2 = \{0\}$ if $M_1 \perp M_2$ and M_1, M_2 are subspaces.

* For <u>Banach</u> spaces, we do not have scalar product, and the notation $M \oplus N$ would have the "old" meaning. (See Rudin 4.20) However, then in an Hilbert space $M \oplus N$ could, in principle, mean three different logical objects. If we ever use $\oplus$ in any other that the Hilbert space setup, we will denote it by "$\oplus_v$" (v = vector space sense.).

* Many thanks for Petri Vesanen for pointing out the error!

* The following relations are then obvious but good to keep in mind when applying the general results derived later:

(a) Banach, Fréchet and F-spaces are complete metric spaces
(b) V Banach $\Rightarrow$ V Fréchet $\Rightarrow$ V F-space
(c) V Fréchet and locally bounded $\Rightarrow$ V Banach.

Proof (a) $\Leftarrow$ definitions. For (b): V Banach $\Rightarrow$ normable $\underset{3.}{\overset{\not}{\Rightarrow}}$ locally convex.

For (c): V Fréchet & loc. bnded $\Rightarrow$ loc. convex & bnded $\underset{3.}{\Rightarrow}$ normable. □

* Note: The Fréchet metric in (c) does not need to be given by a norm. The stament is that there is a norm which yields the same topology, and that V is complete under the norm-metric.
* Note: in (c) "locally bounded" is taken in the sense defined earlier, i.e., "topologically" bounded. This turns out to be the same as "metric-bounded" for norms, but not for generic Fréchet-metrics.

4.1. Addition: Topology reminder:
* Jointly continuous $\Rightarrow$ separately continuous
* If X_1, X_2 are topological spaces, the product topology is the weakest topology s.t. the component projections $P_i: X_1 \times X_2 \to X_i$, $i=1,2$, are continuous. The collection $\{U_1 \times U_2 \mid U_i \subset X_i \text{ open}, i=1,2\}$ is a base for the product topology.
* Thus (b) is equivalent to requiring that: If $U \in T_V$ and $x, y \in V$ s.t. $x+y \in U$, then $\exists U_x, U_y \in T_V$ s.t. $x \in U_x$, $y \in U_y$ and $U_x + U_y \subset U$.
* (c) $\Leftrightarrow$ [if $U \in T_V$, $\alpha \in \mathbb{K}$, $x \in V$ and $\alpha x \in U$, then $\exists \varepsilon > 0$, $U_x \in T_V$ s.t. $x \in U_x$ and $\beta U_x \subset U$ $\forall \beta: |\beta - \alpha| < \varepsilon$]

4.7. Theorem (Separation)

Suppose $K, C \subset \mathcal{V}$, where $\mathcal{V}$ is a topological vector space, K is compact, and C is closed. If $K \cap C = \emptyset$, then there is a neighborhood V of $0 \, (=0_{\mathcal{V}})$ such that

$$(*) \qquad (K+V) \cap (C+V) = \emptyset .$$

* Since $K+V = \bigcup_{x \in K} (x+V)$, it is an open set. (4.4.) and so is $C+V$. Thus the theorem says that a pair of disjoint closed and compact sets have a non-overlapping open cover.

* Before proof, let us state two useful Corollaries:

4.9. Corollary If $\mathcal{B}$ is a local base for a topol. vect. space $\mathcal{V}$, then for any $V \in \mathcal{B}$ there is $U \in \mathcal{B}$ s.t. $\overline{U} \subset V$.

Pf. If $V \in \mathcal{B} \Rightarrow V^c$ is closed. On the other hand, since $0 \in V \Rightarrow \{0\} \cap V^c = \emptyset$. $\{0\}$ is compact, Thrm. 4.7. $\Rightarrow \exists V'$ open s.t. $0 \in V'$ and $V' \cap (V^c + V') = \emptyset$. $\Rightarrow \exists U \in \mathcal{B}$ s.t. $0 \in U \subset V'$, and $U \cap (V^c + U) = \emptyset$. But then $V^c + U \subset U^c$, where $V^c + U$ is open. $\Rightarrow U \subset (V^c+U)^c$ = closed.
$\Rightarrow \overline{U} \subset (V^c + U)^c \subset V$. □
since $0 \in U$.

[Notation: $V^c = \mathcal{V} \setminus V$ = complement of V.]

4.8. Corollary Every topological vector space is a Hausdorff space.

Pf. If $x, y \in \mathcal{V}$, $x \neq y$, then $\{x\}$ is compact and by assumption $\{y\}$ is closed. $\Rightarrow \exists V$ open s.t. $0 \in V$ and $(x+V) \cap (y+V) = \emptyset$. Here $0 \in V \Rightarrow x \in x+V$ = open and $y \in y+V$ = open.
$\therefore \mathcal{V}$ is Hausdorff. □

Proof of Thrm 4.7.

4.10. Lemma If W is a neighborhood of 0 in $\mathcal{V}$, then there is U open s.t. $0 \in U$, $-U = U$, and $U + U \subset W$.

Proof. $0 = 0 + 0$, and since addition is continuous $\exists V_1, V_2$ open s.t. $0 \in V_1$, $0 \in V_2$ and $V_1 + V_2 \subset W$. Define $U = V_1 \cap V_2 \cap (-V_1) \cap (-V_2)$, which is open as $-V_1, -V_2$ are open. (4.3.)
Clearly, $x \in U \Leftrightarrow -x \in U$, and $U + U \subset V_1 + V_2 \subset W$.
Since $-0 = 0$, we have also $0 \in U$. □

If $K = \emptyset$, then $K + V = \emptyset$, and any neighborhood V of 0 satisfies $(*)$.
Assume $K \neq \emptyset$, and consider any $x \in K$. Since C is closed and by assumption $x \notin C$, $\exists$ open V s.t. $0 \in V$ and $x + V \subset C^c$. By 4.10., there is U_1 s.t. $0 \in U_1$, $-U_1 = U_1$, and $U_1 + U_1 \subset V$. But applying 4.8. to U_1 shows that $\exists V_x$ open s.t. $0 \in V_x$, $-V_x = V_x$, and $V_x + V_x \subset U_1$. $\Rightarrow$

$V_x + V_x + V_x + V_x \subset U_1 + U_1 \subset V$. $\overset{0 \in V_x}{\Rightarrow}$

$x + V_x + V_x + V_x \subset x + V \subset C^c$. Then, if $y \in (x + V_x + V_x) \cap (C + V_x)$

$\Rightarrow \exists y_1, y_2, y_3 \in V_x$, $x' \in C$ s.t.

$$y = x + y_1 + y_2 = x' + y_3 \Rightarrow x' = x + y_1 + y_2 \underbrace{- y_3}_{\in V_x}$$

$\in C \cap (x + V_x + V_x + V_x) = \emptyset$.

This is a contradiction, and thus

$(x + V_x + V_x) \cap (C + V_x) = \emptyset$.

As K is compact and $K \subset \bigcup_{x \in K} (x + V_x)$, $\exists$ finitely many $x_j \in K$, $j = 1, \dots, n$, s.t.

$$K \subset \bigcup_{j=1}^{n} (x_j + V_{x_j}). \text{ Let } V = \bigcap_{j=1}^{n} V_{x_j}. \text{ Then}$$

V is open, $0 \in V$, and $K + V \subset \bigcup_{j=1}^{n} (x_j + V_{x_j} + V)$

$\subset \bigcup_{j=1}^{n} (x_j + V_{x_j} + V_{x_j})$. Therefore, $y \in K + V \Rightarrow$

$\exists j$ s.t. $y \in x_j + V_{x_j} + V_{x_j} \Rightarrow y \notin C + V_{x_j} \Rightarrow y \notin C + V$. □

4.11. Proposition. Let $\mathcal{V}$ be a topol. vector space.

(a) If $A \subset \mathcal{V}$, then $\bar{A} = \bigcap_{\substack{V \text{ open,} \\ 0 \in V}} (A+V)$.

(b) If $A, B \subset \mathcal{V}$, then $\bar{A} + \bar{B} \subset \overline{A+B}$

(c) If $\mathcal{Y} \subset \mathcal{V}$ is a subspace, then so is $\bar{\mathcal{Y}}$.

(d) If $A \subset \mathcal{V}$ is convex, so are $\bar{A}$ and A°.
(A° = interior of $A := \bigcup \{ U \subset A \mid U \text{ open} \}$)

(e) If $A \subset \mathcal{V}$ is balanced, so is $\bar{A}$. If also $0 \in A^\circ$, then A° is balanced.

(f) If $A \subset \mathcal{V}$ is bounded, then so is $\bar{A}$.

Proof a) First note that if V is open and $V \cap A = \emptyset$
$\Rightarrow V^c$ is closed and $A \subset V^c \Rightarrow \bar{A} \subset V^c \Rightarrow V \cap \bar{A} = \emptyset$.
Thus if $x \in \bar{A}$ and V is neighb. of 0, then $x - V$ is open and contains an element of $\bar{A}$
$\Rightarrow (x-V) \cap A \neq \emptyset \Rightarrow \exists x_0 \in A$ and $v_0 \in V$ s.t. $x - v_0 = x_0 \Rightarrow x = x_0 + v_0 \in A + V$.
$\therefore \bar{A} \subset \bigcap_{V \text{ nbrh of } 0} (A+V)$.

If $x \notin \bar{A}$, then $\exists V$ open s.t. $0 \in V$ and $x + V \subset \bar{A}^c$ (= open) $\Rightarrow A \cap (x+V) = \emptyset$
$\Rightarrow x \notin A - V$ where $-V$ is open and $0 \in -V$.
$\therefore \bar{A}^c \subset [\bigcap (A+V)]^c$.
Therefore, $\bar{A} = \bigcap_{V:\sim} (A+V)$ ∎

b) Suppose $x \in \bar{A}$, $y \in \bar{B}$, when we need to show that $x+y \in \overline{A+B}$. Let V be open and $0 \in V$. Since $x+y-V$ is open and addition is continuous, (and $x+y \in x+y-V$) $\exists W_x, W_y$ open s.t. $W_x + W_y \subset x+y-V$ and $x \in W_x, y \in W_y$.
But then $W_x \cap A \neq \emptyset$ and $W_y \cap B \neq \emptyset$. $\Rightarrow \exists a \in A, b \in B$ s.t. $a + b \in W_x + W_y \subset x+y-V \Rightarrow \exists v_0 \in V$ s.t. $x+y = a+b+v_0 \in A+B+V$.
By a) this proves $x+y \in \overline{A+B}$. ∎

c) By Proposition 4.3, if $\alpha \in \mathbb{k}, \alpha \neq 0$, then M_α is homeomorphism. Thus then $M_\alpha(\bar{\mathcal{Y}}) = \overline{M_\alpha \mathcal{Y}}$, i.e., $\alpha \bar{\mathcal{Y}} = \overline{\alpha \mathcal{Y}}$. If $\alpha = 0$, we have $0\bar{\mathcal{Y}} = \{0\} = \overline{\alpha \mathcal{Y}}$, since $\{0\}$ is closed. $\therefore \forall \alpha \in \mathbb{k}: \alpha \bar{\mathcal{Y}} = \overline{\alpha \mathcal{Y}}$.

Since $Y \neq \emptyset \Rightarrow \overline{Y} \neq \emptyset$. Also, if $\alpha, \beta \in \Bbbk$, we now have $\alpha\overline{Y} + \beta\overline{Y} = \overline{\alpha Y} + \overline{\beta Y} \underset{\uparrow b)}{\subset} \overline{\alpha Y + \beta Y} \underset{\uparrow Y \text{ subspace}}{\subset} \overline{Y}$

This proves that $\overline{Y}$ is a subspace. ∎

Proof of d), e), f) is left as an exercise (4.4.) □

4.12 Theorem: In a topol. vect. space V

a) every neighborhood of 0 contains a balanced neighborhood of 0;
b) every convex neighborhood of 0 contains a balanced convex neighborhood of 0.

Proof a) Suppose $U \in T_V$ and $0 \in U$. Since $M_{\alpha=0}$ is contin. and $0 0_V = 0_V \in U \Rightarrow \exists \varepsilon > 0$ and $U_0 \in T_V$ s.t. $0 \in U_0$ and $\beta U_0 \subset U$ $\forall \beta \in \Bbbk: |\beta| < \varepsilon$.
If $\beta \neq 0$, $\beta U_0 = M_\beta(U_0)$ is open, and $0 = \beta 0 \in \beta U_0$. Let $V := \bigcup_{0<|\beta|<\varepsilon} (\beta U_0)$. Then V is open, $0 \in V$.

If $\alpha \in \Bbbk$, $0 < |\alpha| \leq 1$, and $x \in V$, then $\exists x_0 \in U_0$, $\beta \in \Bbbk$, s.t. $0 < |\beta| < \varepsilon$ and $x = \beta x_0 \Rightarrow \alpha x = \alpha\beta x_0 \in V$ since $0 < |\alpha\beta| \leq |\beta| < \varepsilon$. For $\alpha = 0$, $\alpha V = \{0\} \subset V$, and thus $\forall |\alpha| \leq 1: \alpha V \subset V$. $\Rightarrow$ V is balanced neighb. of 0 for which $V \subset U$.

b) Suppose $U \in T_V$ is convex and $0 \in U$. Choose ε and U_0 as in a) and define again $V := \bigcup_{0<|\beta|<\varepsilon} (\beta U_0)$.

Let $A = \bigcap_{|\alpha|=1} (\alpha U)$. If $|\alpha| = 1$, we have $|\alpha^{-1}| = 1$,

and thus $\alpha^{-1} V \subset V$ and $\alpha V \subset V \Rightarrow \alpha^{-1} V = V$.
$\Rightarrow V = \alpha(\alpha^{-1} V) = \alpha V \subset \alpha U$. Therefore, $V \subset A$, and thus $V \subset A^\circ \Rightarrow A^\circ$ is a neighb. of 0. Clearly, $A^\circ \subset A \subset 1U = U$. Since each αU, $|\alpha| = 1$, is convex, so is A. By 4.11.d) $\Rightarrow A^\circ$ is convex. Since $0 \in A^\circ$, to prove that A° is balanced, it suffices to prove that A is balanced (4.11.e)

For this, consider $0 \le r \le 1$ and $\beta \in \mathbb{K}$ s.t. $|\beta| = 1$. Then

$$r\beta A = \bigcap_{|\alpha|=1} (r \underbrace{\beta\alpha}_{|\,|=1} U) = \bigcap_{|\alpha|=1} (r\alpha U)$$

Here if $x_0 \in U$ and $|\alpha| = 1 \Rightarrow r\alpha x_0 = \alpha(rx_0 + (1-r)0) \in \alpha U$ since U is convex and $0, x_0 \in U$.

$\Rightarrow r\beta A \subset \bigcap_{|\alpha|=1} (\alpha U) = A$. Since for any $\alpha \in \mathbb{K}$ with $|\alpha| \le 1$, we can find r, β s.t. $\alpha = r\beta$, this proves that A is balanced. $\square$

<u>4.13.</u> Direct corollaries:

a) Every topological vector space has a balanced local base.

b) Every locally convex topol. vect. space has a balanced convex local base.

<u>4.14.</u> Theorem: Let $\mathcal{V}$ be a topol. vect. space, $V_0 \in \mathcal{T}_{\mathcal{V}}$ and $0 \in V_0$.

(a) If $(r_n)_{n\in\mathbb{N}_+}$ is a strictly increasing sequence s.t. $r_n > 0 \ \forall n$, and $r_n \xrightarrow{n\to\infty} \infty$, then

$$\mathcal{V} = \bigcup_{n=1}^{\infty} (r_n V_0)$$

(b) If $K \subset \mathcal{V}$ is compact, then K is bounded.

(c) Let $(\delta_n)_{n\in\mathbb{N}_+}$ be strictly decreasing, s.t. $\delta_n > 0 \ \forall n$, and $\delta_n \xrightarrow{n\to\infty} 0$. If V_0 is <u>bounded</u>, then the collection

$$\mathcal{B} := \{ \delta_n V_0 \mid n \in \mathbb{N}_+ \}$$

is a local base for $\mathcal{V}$.

<u>Proof</u>: a) Fix $x \in \mathcal{V}$. Since the map $\alpha \mapsto \alpha x \in \mathcal{V}$ is continuous, the set $\{\alpha \mid \alpha x \in V_0\} \subset \mathbb{K}$ is open and contains 0. Thus $\exists n_0 \in \mathbb{N}_+$ s.t. $r_n^{-1} x \in V_0 \ \forall n \ge n_0$. In particular, $x \in r_{n_0} V_0$. $\blacksquare$

b) By 4.12 a) $\exists W \subset V_0$ s.t. W is balanced, open and $0 \in W$. By a), $K \subset V = \bigcup_{n=1}^{\infty} (nW)$.

As K is compact, $\exists n_0 \in \mathbb{N}_+$ s.t.
$K \subset \bigcup_{n=1}^{n_0} (nW) = n_0 W$. ($n \leq n_0$, $x \in nW \Rightarrow \exists x_0 \in W$ s.t. $x = n x_0 = n_0 \frac{n}{n_0} x_0 \underset{\uparrow \text{W balanced.}}{\in} n_0 W$.)

Therefore, if $t > n_0$ and $x \in K \Rightarrow x \in n_0 W = t \frac{n_0}{t} W \subset tW$, as W is balanced.
$\therefore$ $K \subset tV_0$. Since V_0 was an arbitrary neighb. of 0, this proves that K is bounded. ■

c) Consider $U \in \mathcal{T}_V$ s.t. $0 \in U$. If V_0 is bounded, $\exists s > 0$ s.t. $V_0 \subset tU$ $\forall t > s$. Then $\exists n_0 \in \mathbb{N}_+$ s.t. $s\delta_n < 1$ $\forall n \geq n_0$. ($s\delta_n \to 0^+$.) $\Rightarrow V_0 \subset \delta_n^{-1} U$ $\forall n \geq n_0$.
$\Rightarrow \delta_{n_0} V_0 \subset U$. Since each $\delta_n V_0$ is open and contains 0, this proves that $\mathcal{B}$ is a local base. □