3. Bounded operators: $\mathcal{B}(\mathcal{H})$

* As in the general case in 2.2., we define for $V = \mathcal{H}$ the space

$$\mathcal{B}(\mathcal{H}) = \{ T : \mathcal{H} \to \mathcal{H} \mid T \text{ linear and } \|T\| < \infty \}$$

with $\|T\| = \sup \{ \|T\psi\| \mid \psi \in \mathcal{H}, \|\psi\| = 1 \}$.

* An operator on $\mathcal{H}$ is a linear mapping

$A : D \to \mathcal{H}$, with $D \subset \mathcal{H}$ subspace.

$D = D(A)$ = domain of A.
$R(A) = \{ A\psi \mid \psi \in D \}$ = range of A.
$\mathrm{Ker}(A) = \{ \psi \in D \mid A\psi = 0 \}$ = null space or kernel of A

* $\mathcal{B}(\mathcal{H})$ is also called the set of bounded operators. Note that $T \in \mathcal{B}(\mathcal{H})$ implies $D(T) = \mathcal{H}$.

* $\mathcal{B}(\mathcal{H}) = \{$ the set of contiuous linear transformations of $\mathcal{H} \}$ (~~by Ex. 1.1~~) Proof later.

* $\mathcal{B}(\mathcal{H})$ is a Banach space, since $\mathcal{H}$ is complete. (Proof later.)

* The following yields an important classification of bounded linear and sesquilinear functionals on $\mathcal{H}$:

3.1. Thm: a) Suppose $\Lambda : \mathcal{H} \to \mathbb{C}$ is linear and bounded. Then $\exists!\ \psi_0 \in \mathcal{H}$ s.t.

$$\Lambda\psi = (\psi_0, \psi) \quad \forall \psi \in \mathcal{H}.$$

b) Suppose $\Gamma : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is bounded and sesquilinear; that is, assume that

(i) $\Gamma(\phi, \alpha\psi_1 + \beta\psi_2) = \alpha\Gamma(\phi,\psi_1) + \beta\Gamma(\phi,\psi_2)$,
$\Gamma(\alpha\phi_1 + \beta\phi_2, \psi) = \alpha^*\Gamma(\phi_1,\psi) + \beta^*\Gamma(\phi_2,\psi)$
$\forall\, \phi, \phi_1, \phi_2, \psi, \psi_1, \psi_2 \in \mathcal{H}$, $\alpha, \beta \in \mathbb{C}$.

(ii) $\exists\, C \geq 0$ s.t. $|\Gamma(\phi,\psi)| \leq C\|\phi\|\,\|\psi\|$ $\forall \phi,\psi \in \mathcal{H}$.

Then $\exists!\, T \in \mathcal{B}(\mathcal{H})$ s.t. $\forall \phi, \psi \in \mathcal{H}$

$$\Gamma(\phi,\psi) = (\phi, T\psi)$$

and $\|T\| = \sup\{\, |\Gamma(\phi,\psi)| \;|\; \|\phi\| = 1 = \|\psi\|\} \leq C$.

<u>pf:</u> a) Boundedness of Λ means that (compare to **2.2.**)
$\|\Lambda\| := \sup\{\, |\Lambda\psi| \;|\; \psi \in \mathcal{H}, \|\psi\| = 1\} < \infty$.
Let us start with uniqueness: If $\psi_0, \psi_0' \in \mathcal{H}$ s.t. $(\psi_0, \psi) = (\psi_0', \psi)$ $\forall\psi \Rightarrow$
$(\psi_0 - \psi_0', \psi) = 0$ $\forall\psi \Rightarrow$
$0 = (\psi_0 - \psi_0', \psi_0 - \psi_0') = \|\psi_0 - \psi_0'\|^2$
$\Rightarrow \psi_0' = \psi_0$. thus ψ_0 is unique.
If $\Lambda = 0 \Rightarrow \Lambda\psi = 0 = (0,\psi)$ $\forall\psi \Rightarrow \psi_0 = 0$ is o.k. If $\Lambda \neq 0$, let $M = \ker(\Lambda)$ $:= \{\psi \in \mathcal{H} \mid \Lambda\psi = 0\}$. Since Λ is bounded $\Rightarrow \Lambda$ continuous $\Rightarrow M = \Lambda^{\leftarrow}(\{0\})$ is closed. It is also obviously a subspace. By Thm. 2.11 $\mathcal{H} = M \oplus M^\perp$. Now $M^\perp \neq \{0\}$ since else $M = \mathcal{H}$ which would mean $\Lambda = 0$. Thus $\exists\, \phi \in M^\perp$, $\phi \neq 0$.
However, then for any $\psi \in \mathcal{H}$
$\Lambda((\Lambda\psi)\phi - (\Lambda\phi)\psi) = (\Lambda\psi)(\Lambda\phi) - (\Lambda\phi)(\Lambda\psi) = 0$
$\Rightarrow (\Lambda\psi)\phi - (\Lambda\phi)\psi \in M$. Then by $\phi \in M^\perp$
$\Rightarrow (\Lambda\psi)\underbrace{(\phi,\phi)}_{\neq 0} - (\Lambda\phi)(\phi,\psi) = 0$

$$\Rightarrow \Lambda\psi = \frac{\Lambda\phi}{\|\phi\|^2}(\phi,\psi) = (\psi_0, \psi)$$

for $\psi_0 = \dfrac{(\Lambda\phi)^*}{\|\phi\|^2}\phi$. This proves a). ◢

b) Is a corollary of a): For any $\psi \in \mathcal{H}$ by (i),(ii) the map $\Lambda_\psi : \phi \mapsto \Gamma(\phi,\psi)^*$ is linear, and $|\Lambda_\psi\phi| \leq C\|\psi\|$ if $\|\phi\| = 1$.

$\dashrightarrow$

Thus by a), $\exists!\ \psi_0 \in \mathcal{H}$ s.t. $\Lambda_\psi \phi = (\psi_0, \phi)\ \forall \phi \in \mathcal{H}$.
We denote the map $\psi \mapsto \psi_0$ by T, when

$$\forall \phi, \psi \in \mathcal{H} : \quad (T\psi, \phi) = \Lambda_\psi \phi = \Gamma(\phi, \psi)^*$$
$$\Rightarrow \quad \Gamma(\phi, \psi) = (\phi, T\psi).$$

By linearity of $\Gamma(\phi, \cdot) \Rightarrow$

$$\forall \phi \in \mathcal{H} : \quad (\phi, T(\alpha\psi_1 + \beta\psi_2)) = \Gamma(\phi, \alpha\psi_1 + \beta\psi_2)$$
$$= \alpha(\phi, T\psi_1) + \beta(\phi, T\psi_2)$$
$$\Rightarrow \quad T(\alpha\psi_1 + \beta\psi_2) = \alpha T\psi_1 + \beta T\psi_2 .$$

Thus T is linear. Also

$$\|T\psi\|^2 = (T\psi, T\psi) = \Gamma(T\psi, \psi) \leq C\, \|T\psi\|\, \|\psi\|.$$
$$\Rightarrow \text{ if } \|\psi\| \leq 1, \quad \|T\psi\| \leq C \Rightarrow \|T\| \leq C < \infty.$$

$\Rightarrow T \in \mathcal{B}(\mathcal{H})$ and, as for any $\phi \in \mathcal{H}$

$$\|\phi\| = \sup\{ |(\phi', \phi)| \mid \phi' \in \mathcal{H},\ \|\phi'\| = 1\}$$

(Proof: Cauchy-Schwarz),

we also have

$$\|T\| = \sup\{ \|T\psi\| \mid \|\psi\| = 1\}$$
$$= \sup\{ |(\phi, T\psi)| \mid \|\psi\| = 1 = \|\phi\|\}$$
$$= \sup\{ |\Gamma(\phi, \psi)| \mid \|\psi\| = 1 = \|\phi\|\} \leq C.$$

For uniqueness: If $T' \in \mathcal{B}(\mathcal{H})$ is s.t.

$$\Gamma(\phi, \psi) = (\phi, T'\psi) \quad \forall \phi, \psi \Rightarrow$$
$$0 = (\phi, T'\psi - T\psi) \quad \forall \phi, \psi \Rightarrow T'\psi = T\psi \quad \forall \psi$$
$$\Rightarrow T' = T. \quad \square$$

* a) is called Riesz Lemma. (Not to be confused with the other Riesz-theorems: Riesz-Fischer, ...)

3.2. Adjoint of a bounded operator

If $T \in \mathcal{B}(\mathcal{H})$, for all $\phi, \psi \in \mathcal{H}$

$$|(\phi, T\psi)| \overset{C.-S.}{\leq} \|\phi\|\, \|T\psi\| \leq \|T\|\, \|\phi\|\, \|\psi\|$$

since $\|T\psi\| \leq \|T\|\, \|\psi\|$. (If $\psi = 0 \Rightarrow T\psi = 0$ $\Rightarrow \|T\psi\| = 0 = \|T\|\, \|\psi\|$. Else $\|T\psi\| = \|T \underbrace{\tfrac{\psi}{\|\psi\|}}_{\|\cdot\| = 1}\|\, \|\psi\|$ $\leq \|T\|\, \|\psi\|$, by definition of $\|T\|$.)

therefore, $\Gamma(\phi, \psi) = (T\phi, \psi) = (\psi, T\phi)^*$ satisfies the assumptions of Th. 3.1. b).

$\Rightarrow \exists!\, T^* \in \mathcal{B}(\mathcal{H})$ s.t. $(T\phi, \psi) = (\phi, T^*\psi)$ $\forall \phi, \psi$.

Also it follows that $\|T^*\| = \{ |(T\phi,\psi)| \mid \|\phi\| = 1 = \|\psi\| \}$
$= \{ |(\psi, T\phi)| \mid \|\phi\| = 1 = \|\psi\| \} = \|T\|$.

* The operator T^* is called the adjoint of T.

* The adjoint mapping $T \mapsto T^*$ defines an involution ($(T^*)^* = T$) on $\mathcal{B}(\mathcal{H})$ which makes it into a C^*-algebra:

3.3. Thm: For all $T, S \in \mathcal{B}(\mathcal{H})$, $\alpha \in \mathbb{C}$

a) $(T+S)^* = T^* + S^*$
b) $(\alpha T)^* = \alpha^* T^*$
c) $(ST)^* = T^* S^*$
d) $T^{**} = T$
e) $\|T^* T\| = \|T\|^2$.

(Notations: $ST := S \circ T$ and $T^{**} := (T^*)^*$)

Pf. Exercise □

3.4. Definitions

An operator $T \in \mathcal{B}(\mathcal{H})$ is called

a) self-adjoint if $T^* = T$
b) unitary if $T^*T = 1 = TT^*$
($\Rightarrow$ unitary also in the general Hilbert-space-isomorphism-sense)
c) normal if $T^*T = TT^*$
d) projection if $T^2 = T$.
If also $R(T) = \ker(T)^\perp$, T is called an orthogonal projection.

* Note: P and Q in Theorem 2.16. are orthogonal projections.

1 * $V \subset S$ is a neighborhood of $p \in S$ if $V \in \mathcal{T}_S$ and $p \in V$.

* A local base at $p \in S$ is a collection $\mathcal{B}_p$ of neighborhoods of p s.t. any neighborhood of p contains a member of $\mathcal{B}_p$. In formulae: $\mathcal{B}_p \subset \mathcal{T}_S$, $U \in \mathcal{B}_p \Rightarrow p \in U$, and $V \in \mathcal{T}_S$, $p \in V \Rightarrow \exists U \in \mathcal{B}_p$ s.t. $U \subset V$.

3.5. Various topologies on $\mathcal{H}$ and $\mathcal{B}(\mathcal{H})$

* Topology reminder: Suppose $\mathfrak{X}$ is a topological space and S is a set. Then any non-empty collection $\mathcal{K}$ of functions $S \to \mathfrak{X}$ defines a topology on S via the following construction:

1) Consider the inverse images of open sets:
$$\mathcal{b} := \{ f^{\leftarrow}(U) \mid f \in \mathcal{K},\ U \circledcirc \mathfrak{X} \}$$
2) Let $\mathcal{B}$ collect all finite intersections of elements in $\mathcal{b}$
$$\mathcal{B} := \{ \bigcap_{i=1}^{n} V_i \mid \text{for some } n,\ V_i \in \mathcal{b}\ \forall i \}$$
3) The topology $\mathcal{T}_{\mathcal{K}}$ on S consists of all unions of elements of $\mathcal{B}$.

* $\mathcal{T}_{\mathcal{K}}$ is the topology generated by $\mathcal{b}$, and $\mathcal{B}$ forms a base for $\mathcal{T}_{\mathcal{K}}$.

1 In general, we recall that:

* $T_{\mathcal{K}}$ is called the <u>$\mathcal{K}$-weak topology</u>

since it is the <u>weakest</u> topology on S, for which all $f \in \mathcal{K}$ are continuous.

<u>Examples</u>:

a) <u>Weak topology on $\mathcal{H}$</u>
= $\mathcal{K}$-weak topology with $\mathfrak{X} = \mathbb{C}$ (with the standard topology) and
$\mathcal{K} := \{ \Lambda_\phi : \mathcal{H} \to \mathbb{C} \mid \phi \in \mathcal{H} \}$,
where $\Lambda_\phi(\psi) := (\phi, \psi)$.

b) <u>Strong operator topology on $\mathcal{B}(\mathcal{H})$</u>

= $\mathcal{K}$-weak topology with $\mathfrak{X} = \mathcal{H}$ (with the norm-topology), and
$\mathcal{K} := \{ E_\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{H} \mid \phi \in \mathcal{H} \}$
where $E_\phi(T) := T\phi$.

c) <u>weak operator topology on $\mathcal{B}(\mathcal{H})$</u>

= $\mathcal{K}$-weak topology with $\mathfrak{X} = \mathbb{C}$ (stand. topol.) and

$$\mathcal{K} := \{ \Gamma_{\phi,\psi} : \mathcal{B}(\mathcal{H}) \to \mathbb{C} \mid \phi, \psi \in \mathcal{H} \}$$

where $\Gamma_{\phi,\psi}(T) := (\phi, T\psi)$.

<u>Remarks</u>: * To make matters confusing, there is also a <u>weak topology</u> on $\mathcal{B}(\mathcal{H})$, which is induced by
$\mathcal{K} := \{ \Lambda : \mathcal{B}(\mathcal{H}) \to \mathbb{C} \mid \Lambda \text{ continuous} \}$.
Since every $\Gamma_{\phi,\psi} \in \mathcal{K}$, the weak $\mathcal{B}(\mathcal{H})$-topology is stronger than the weak operator topology.

* The above topologies are distinct from the norm-topologies, if $\dim \mathcal{H} = \infty$. (See exercises)

4. Topological vector spaces

4.1. Definition: V is called a topological vector space if

(a) It is a vector space with topology $\mathcal{T}_V$
(b) Addition is continuous as a map $V \times V \to V$
(c) Scalar multiplication is continuous as a map $\mathbb{K} \times V \to V$
($\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ = scalar field)
(d) For any $x \in V$ the set $\{x\}$ is closed.
(Singlets are closed sets.)

* From now on we assume V is a topol. vect. space.

4.2. Terminology and notations:

* For $x \in V$, $A, B \subset V$; $\alpha \in \mathbb{K}$,

$$x + A := \{x + y' \mid y' \in A\}$$
$$A + B := \{x' + y' \mid x', y' \in A\}$$
$$\alpha A := \{\alpha x' \mid x' \in A\}$$

* The following terminology is applied to subsets $A \subset V$: A is called

as before { (a) subspace $\Leftrightarrow$ $A \neq \emptyset$ and $\forall \alpha, \beta \in \mathbb{K}: \alpha A + \beta A \subset A$
(b) convex $\Leftrightarrow$ $\forall t \in [0,1]: tA + (1-t)A \subset A$ }
(c) balanced $\Leftrightarrow$ $\forall \alpha \in \mathbb{K}$ with $|\alpha| \leq 1$: $\alpha A \subset A$
(d) bounded $\Leftrightarrow$ For any $V \in \mathcal{T}_V$ with $0 \in V$: $\exists s > 0$ s.t. $A \subset tV$ $\forall t > s$.
(i.e., A is contained in any sufficiently "dilated" neighbourhood of zero.)

* Warning: not all "usual" relations hold for subsets. For instance, it may happen that $2A \neq A + A$.

4.3. Define:

(a) For $x_0 \in V$ the translation operator

$$T_{x_0} : V \to V \quad \text{by} \quad T_{x_0}(x) := x_0 + x \quad \forall x \in V.$$

(b) For $\alpha \in \mathbb{k}$ the scalar multiplication operator

$$M_\alpha : V \to V \quad \text{by} \quad M_\alpha(x) := \alpha x \quad \forall x \in V.$$

Proposition: T_{x_0}, $x_0 \in V$, and M_α, $\alpha \neq 0$, are homeomorphisms.

Proof. Since V is a vector space, it easily follows that both maps are bijective and that $T_{x_0}^{-1} = T_{-x_0}$ and $M_\alpha^{-1} = M_{\frac{1}{\alpha}}$. On the other hand, any T_x and $M_{\alpha'}$, $x \in V$, $\alpha' \in \mathbb{k}$, is continuous as V is a topol. vect. space. □

Corollary: $\mathcal{T}_V$ is translation-invariant:

$$U \subset V \text{ is open } \underline{\text{iff}} \quad x + U \text{ is open } \forall x \in V.$$

4.4. Definition: $\mathcal{B} \subset \mathcal{T}_V$ is called a local base if it is (topologically) a local base at $0 \in V$.

* By the Corollary in 4.3. a local base $\mathcal{B}$ determines $\mathcal{T}_V$: If $U \in \mathcal{T}_V$ and $x_0 \in U$, then $0 \in T_{-x_0}(U) = -x_0 + U$ = open $\Rightarrow$ $\exists U_0 \in \mathcal{B}$ s.t. $U_0 \subset -x_0 + U \Rightarrow x_0 + U_0 \subset U$. Thus $\{x_0 + U_0\}_{U_0 \in \mathcal{B}}$ forms a local base at x_0.
$\therefore$ $U \in \mathcal{T}_V \Leftrightarrow U$ = union of translates of $U_0 \in \mathcal{B}$.

4.5. Definition: If $d : V \times V \to \mathbb{R}$ is a metric, it is called (translation-)invariant if

$$d(x + x', y + x') = d(x, y) \quad \forall x, y, x' \in V.$$

* Note: $\mathcal{T}_V$ does not need to be given by a metric.

4.5. Terminology: $\mathcal{V}$ is called

(a) locally convex $\Leftrightarrow$ $\exists$ local base $\mathcal{B}$ s.t. each $U_0 \in \mathcal{B}$ is convex

(b) locally bounded $\Leftrightarrow$ $\exists V_0 \in \mathcal{T}_{\mathcal{V}}$ s.t. $0 \in V_0$ and V_0 is bounded

(c) locally compact $\Leftrightarrow$ $\exists V_0 \in \mathcal{T}_{\mathcal{V}}$ s.t. $0 \in V_0$ and $\overline{V_0}$ is compact.

(d) metrizable $\Leftrightarrow$ If $\exists$ metric $d: \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ s.t. $\mathcal{T}_{\mathcal{V}}$ is induced by d.

(Rudin:) (e) F-space $\Leftrightarrow$ If $\mathcal{T}_{\mathcal{V}}$ is induced by a <u>complete</u> <u>invariant</u> metric.

(f) Fréchet space $\Leftrightarrow$ $\mathcal{V}$ is a locally convex F-space (a) & e))

(g) normable $\Leftrightarrow$ $\exists$ norm $\mathcal{V} \to \mathbb{R}$ s.t. $\mathcal{T}_{\mathcal{V}}$ is induced by the norm-metric.

(h) Banach space $\Leftrightarrow$ (g) & completeness.

* In addition, $\mathcal{V}$ is said to have the <u>Heine-Borel property</u> if

$$C \subset \mathcal{V},\ C \text{ is closed and bounded} \Rightarrow C \text{ is compact}$$

4.6. Summary of relations proved in Rudin, FA, Chap. 1: (we will prove some of these later)

1. $\mathcal{V}$ locally bounded $\Rightarrow$ $\mathcal{V}$ has a <u>countable</u> local base
2. $\mathcal{V}$ metrizable $\Leftrightarrow$ $\mathcal{V}$ has a countable local base
3. $\mathcal{V}$ normable $\Leftrightarrow$ $\mathcal{V}$ is locally convex and locally bounded
4. $\mathcal{V}$ is finite-dimensional $\Leftrightarrow$ $\mathcal{V}$ is locally compact
5. $\mathcal{V}$ is locally bounded and has the Heine-Borel property $\Leftrightarrow$ $\mathcal{V}$ is finite-dimensional $\Leftrightarrow$ $\mathcal{V} \cong \mathbb{K}^n$, $n = \dim \mathcal{V}$. ($\mathcal{V}$ & $\mathbb{K}^n$ homeomorphic)

* The following relations are then obvious but good to keep in mind when applying the general results derived later:

(a) Banach, Fréchet and F-spaces are complete metric spaces
(b) V Banach $\Rightarrow$ V Fréchet $\Rightarrow$ V F-space
(c) V Fréchet and locally bounded $\Rightarrow$ V Banach.

<u>Proof</u> (a) $\Leftarrow$ definitions. For (b): V Banach $\Rightarrow$ normable $\underset{3.}{\overset{\Leftarrow}{\Rightarrow}}$ locally convex.

For (c): V Fréchet & loc. bnded $\Rightarrow$ loc. convex & bnded $\underset{3.}{\Rightarrow}$ normable. $\square$

* Note: The Fréchet metric in (c) does not need to be given by a norm. The stament is that there is a norm which yields the same topology, and that V is complete under the norm-metric.
* Note: in (c) "locally bounded" is taken in the sense defined earlier, i.e., "topologically" bounded. This turns out to be the same as "metric-bounded" for norms, but not for generic Fréchet-metrics.