2.19 Tensor products of Hilbert spaces

* Unlike direct sums, infinite tensor products are quite strange beasts and we do not need them here. Hence:

Def. Let $N \in \mathbb{N}_+$, $N \geq 2$, and assume that for all $k = 1, \dots, N$, $\mathcal{H}_k$ is a Hilbert space.

For any collection $(\varphi_k) \in \prod_{k=1}^{N} \mathcal{H}_k$, let $\bigotimes_{k=1}^{N} \varphi_k$ denote the map $\prod_{k=1}^{N} \mathcal{H}_k \to \mathbb{C}$ defined by $\left(\bigotimes_{k=1}^{N} \varphi_k\right)(\psi_1, \dots, \psi_N) := \prod_{k=1}^{N} (\psi_k, \varphi_k)_{\mathcal{H}_k}$.

Let $\mathcal{V} := \{ T : \prod_{k=1}^{N} \mathcal{H}_k \to \mathbb{C} \mid \exists M \in \mathbb{N}_+$, and

$\alpha^{(i)} \in \mathbb{C}$, $\varphi_k^{(i)} \in \mathcal{H}_k$, $k = 1, \dots, N$, $i = 1, \dots, M$ s.t.
$T(\psi) = \sum_{i=1}^{M} \alpha^{(i)} \prod_{k=1}^{N} (\psi_k, \varphi_k^{(i)})$ $\forall \psi \in \prod \mathcal{H}_k \}$

$= \{$ finite linear combinations of $\bigotimes_{k=1}^{N} \varphi_k \}$

* Each $\bigotimes_{k=1}^{N} \varphi_k$ is obviously conjugate-multilinear: they are conjugate linear in each component. Therefore, so is any $v \in \mathcal{V}$.

* $\mathcal{V}$ is a vector space under the usual definition of addition and scalar multiplication:
$(\alpha v_1 + \beta v_2)(\psi) := \alpha v_1(\psi) + \beta v_2(\psi)$ $(\in \mathbb{C})$

* Let 0_k denote the null vector of $\mathcal{H}_k$. Then $0_{\mathcal{V}} := \bigotimes_{k=1}^{N} 0_k$ is the null vector of $\mathcal{V}$.

* Note that $\varphi_k \neq \zeta_k$ for some k does not imply that $\otimes \varphi_k \neq \otimes \zeta_k$. For instance, if any of $\varphi_k = 0$, then $\bigotimes_{k=1}^{N} \varphi_k = 0_{\mathcal{V}}$

* For $T_1, T_2 \in \mathcal{V}$ we can write

$$(*) \quad T_\ell = \sum_{i=1}^{M_\ell} \alpha_\ell^{(i)} \bigotimes_{k=1}^{N} \varphi_{k,\ell}^{(i)} \quad ; \ \ell = 1,2$$

Consider $"(T_1, T_2)" = \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} (\alpha_1^{(i)})^* \alpha_2^{(j)}$

$$\cdot \prod_{k=1}^{N} (\varphi_{k,1}^{(i)}, \varphi_{k,2}^{(j)})_{\mathcal{H}_k} \in \mathbb{C}.$$

Claim 1: $"(T_2, T_1)" = ("(T_1, T_2)")^*$

Proof: obvious.

Claim 2: (T_1, T_2) does not depend on the choices made in $(*)$.

Proof. By claim 1, it suffices to consider equivalent representations of T_2 for fixed representation of T_1. Since $"(T_1, T_2)"$

$$= \sum_{i=1}^{M_1} (\alpha_1^{(i)})^* \left(\sum_{j=1}^{M_2} \alpha_2^{(j)} \left(\bigotimes_{k=1}^{N} \varphi_{k,2}^{(j)}\right) (\varphi_{1,1}^{(i)}, \dots, \varphi_{N,1}^{(i)}) \right)$$

$$= \sum_{i=1}^{M_1} (\alpha_1^{(i)})^* \, T_2(\varphi_{1,1}^{(i)}, \dots, \varphi_{N,1}^{(i)})$$

it only depends on the map T_2, not on its representation. $\square$

Thus $(\cdot, \cdot)$ defines a map $\mathcal{V} \times \mathcal{V} \to \mathbb{C}$, and this map is obviously sesquilinear.

Claim 3 $(\cdot, \cdot)$ is a scalar product on $\mathcal{V}$.

Proof: Let $(e_k^{(\ell)})_{\ell \in I_k}$ be an ONB for $\mathcal{H}_k$...

$$\Rightarrow (\varphi_{k,1}^{(i)}, \varphi_{k,1}^{(j)})_{\mathcal{H}_k} = \sum_{\ell \in I_k} (\varphi_{k,1}^{(i)}, e_k^{(\ell)})(e_k^{(\ell)}, \varphi_{k,1}^{(j)})$$

and the sum can be understood as an integral over the counting measure on I_k. Thus

$$(T_1, T_1) = \sum_{i,j=1}^{M_1} (\alpha_1^{(i)})^* \alpha_1^{(j)} \prod_{k=1}^{N} (\varphi_{k,1}^{(i)}, \varphi_{k,1}^{(j)})$$

$$= \sum_{\ell_1 \in I_1} \cdots \sum_{\ell \in I_k} \sum_{i,j=1}^{M_1} (\alpha_1^{(i)})^* \alpha_1^{(j)} \prod_{k=1}^{N} \left[(e_k^{(\ell_k)}, \varphi_{k,1}^{(i)})^* \cdot (e_k^{(\ell_k)}, \varphi_{k,1}^{(j)}) \right]$$

$$= \sum_{(\ell_k \in I_k)_{k=1}^{N}} \Big| \underbrace{\sum_{i=1}^{M_1} \alpha_1^{(i)} \prod_{k=1}^{N} (e_k^{(\ell_k)}, \varphi_{k,1}^{(i)})}_{= T_1((e_k^{(\ell_k)}))} \Big|^2 \geq 0.$$

... Thus $(T_1, T_1) \geq 0$ and $(T_1, T_1) = 0$ iff
$T_1((e_k^{(l_k)})) = 0 \quad \forall\, l \in \prod_{k=1}^N I_k$.
But always,

$$T_1(\mathcal{H}) = \sum_{i=1}^{M_1} \alpha_1^{(i)} \prod_{k=1}^N (\mathcal{H}_k, \varphi_{k,1}^{(i)})$$

$$= \sum_{l \in \prod_{k=1}^N I_k} \sum_{i=1}^{M_1} \alpha_1^{(i)} \prod_{k=1}^N \left[(\mathcal{H}_k, e_k^{(l_k)}) (e_k^{(l_k)}, \varphi_{k,1}^{(i)}) \right]$$

$$= \sum_{l \in \prod_{k=1}^N I_k} \left(\bigotimes_{k=1}^N e_k^{(l_k)}\right)(\mathcal{H})\, T_1((e_k^{(l_k)})_{k=1}^N)$$

which shows that $(T_1, T_1) = 0 \iff T_1 = 0$.
$\therefore$ $(\cdot,\cdot)$ is a scalar product. $\square$

Definition The abstract completion of $(\mathcal{N}, (\cdot,\cdot))$ into a Hilbert space is called the tensor product of $(\mathcal{H}_k)_{k=1}^N$ and it is denoted by $\bigotimes_{k=1}^N \mathcal{H}_k$.

Proposition If $(e_k^{(l)})_{l \in I_k}$ if an ONB of $\mathcal{H}_k$ $\forall k = 1, \dots, N$, then $(\underbrace{\bigotimes_{k=1}^N e_k^{(l_k)}}_{=: e(l)})_{l \in \prod_{k=1}^N I_k}$ is an ONB of $\bigotimes_{k=1}^N \mathcal{H}_k$.

Proof. If $l', l \in \underbrace{\prod_{k=1}^N I_k}_{=: I}$, then $(e(l'), e(l)) = \prod_{k=1}^N \underbrace{(e_k^{(l'_k)}, e_k^{(l_k)})_{\mathcal{H}_k}}_{= \mathbb{1}(l'_k = l_k)}$

$$= \begin{cases} 0, & \text{if } l' \neq l \\ 1, & \text{if } l' = l. \end{cases}$$

$\Rightarrow$ The set $(e(l))_{l \in I}$ is orthonormal. To prove that it is ONB of $\bigotimes_k \mathcal{H}_k$, it suffices to show that $\mathcal{N} \subset \overline{\operatorname{span}\{e(l)\}_l}$, since then
$\bigotimes_k \mathcal{H}_k = \overline{\mathcal{N}} \subset \overline{\operatorname{span}\{e(l)\}} \subset \bigotimes_k \mathcal{H}_k \Rightarrow \overline{\operatorname{span}\{e(l)\}} = \bigotimes_k \mathcal{H}_k$.
Let $T \in \mathcal{N}$ be an arbitrary representative, as before. By the previous proof, we have $\forall \mathcal{H} \in \prod_k \mathcal{H}_k$

$$T(\mathcal{H}) = \sum_{l \in I} e(l)[\mathcal{H}]\, \gamma_l(T), \quad \text{where } \gamma_l(T) := T(e(l))$$

and $\|T\|^2 = \sum_{l \in I} |\gamma_l(T)|^2 < \infty$. Thus the index set $I_T := \{ l \in I \mid \gamma_l(T) \neq 0 \}$ is countable and..

... there is sequence of <u>finite</u> subsets $I^{(n)}$ of I_T s.t. $\lim_{n\to\infty} \sum_{\ell \in I \setminus I^{(n)}} |\gamma_\ell(T)|^2 = 0$. & $I^{(n)} \subset I^{(n+1)}$.

Define $\tau^{(n)} := \sum_{\ell \in I^{(n)}} \gamma_\ell(T)\, e(\ell) \in \operatorname{span}\{e(\ell)\}$

Now T is the $\otimes \mathcal{H}_k$-norm limit of $\tau^{(n)}$, i.e., $T = \sum_{\ell \in I} \gamma_\ell(T) e(\ell)$ also in norm, not only pointwise. (Pf. For any countable subset $J \subset I$ $\|\sum_{\ell \in J} \gamma_\ell e(\ell)\|^2 = \sum_{\ell \in J} |\gamma_\ell|^2$, since the scalar product is continuous, and $e(\ell)$ are orthonormal. Thus, if $\sum_{\ell \in J} |\gamma_\ell|^2 < \infty$, the sum $\sum_{\ell \in J} \gamma_\ell e(\ell)$ converges.

therefore, $\|T - \tau^{(n)}\|^2 = \sum_{\ell \in I \setminus I^{(n)}} |\gamma_\ell(T)|^2 \to 0$.)

this proves that $\mathcal{V} \subset \overline{\operatorname{span}\{e(\ell)\}}$. □

* Note that any element $T \in \otimes \mathcal{H}_k$ defines a uniq. conjugate-multi-linear map on $\Pi_k \mathcal{H}_k$ by the formula $(\psi_k)_{k=1}^N \mapsto (\bigotimes_{k=1}^N \psi_k, T)$, and that this map is equal to the previous def. of T for all $T \in \mathcal{V}$.
If we talk about a multilinear map associated with a vector in $\otimes \mathcal{H}_k$, it refers to this map.

* Note that by construction any $T \in \otimes \mathcal{H}_k = \overline{\mathcal{V}}$ is a norm-limit of a seq. of the simpler maps $T_n \in \mathcal{V}$, and thus is true also "pointwise"!!

$$T(\Psi) = \lim_{n\to\infty} T_n(\Psi). \quad \forall \Psi \in \Pi_k \mathcal{H}_k$$

(Continuity of scalar product.)

* When does $T: \Pi_k \mathcal{H}_k \to \mathbb{C}$ which is conj.-multi-linear belong to $\otimes_k \mathcal{H}_k$? A: Iff. T is separately continuous in each argument and $\sum_{\ell \in I} |T(e_1^{(\ell_1)}, \dots, e_N^{(\ell_N)})|^2 < \infty$ for some collection of ONB's. (Pf. exercise.)

* The construction is not quite as crazy as it looks like: Examples of isomorphisms

a) $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \cong L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$
b) $L^2(I^d) \otimes L^2(I^{d'}) \cong L^2(I^{d+d'})$,
for I = closed interval, $d, d' \in \mathbb{N}_+$
c)
$$\bigotimes_{k=1}^{N} L^2(\mathbb{R}^d) \cong L^2(\mathbb{R}^{Nd}), \quad d, N \in \mathbb{N}_+$$

<u>Proofs</u> a), b) for $I = [0,1]$ are Exercise 2.5. For c), prove that $\otimes$ is associative and use the general construction in Sec. II.4. of [RS I]. $\square$

2.20. Fock spaces

* In both classical and quantum mechanics, the definition of dynamics is typically given for "<u>closed</u>" systems with fixed number of particles.

  What should be done if the number particles inside the system can change?

  Note that this <u>question</u> arises even for systems where the <u>total</u> number of particles is conserved, as soon as we consider dynamics inside a bounded region V of space: particles moving into and away from the region lead to changes in the number of particles in V.

* In quantum mechanics, a natural description of the dynamics is then extending N-particle Hilbert spaces into a Fock space.

Definition (Fock space)

For $N = 1, 2, \dots$, assume that the N-particle dynamics is described by evolution of "wavevectors" in a Hilbert space $\mathcal{H}_N$. The corresponding Fock space is the Hilbert space

$$\mathcal{H}^{(F)} := \bigoplus_{N=0}^{\infty} \mathcal{H}_N$$

where $\mathcal{H}_0 := \mathbb{C}$.

* $\mathcal{H}_0$ is called the vacuum sector, and the vector $\Omega := (1, 0, 0, \dots) \in \mathcal{H}^{(F)}$ is called the vacuum vector.
  ($\mathcal{H}_0$ is a placeholder for 0-particle states.)

* Recall that the probabilistic interpretation of QM requires that wave-vectors ~~are~~ have a unit norm. The same is true for vectors in the Fock space, after we make the following physical "interpretation":

By def., if $\Psi \in \mathcal{H}^{(F)}$ with $\|\Psi\| = 1$,

we have $\quad 1 = \sum_{N=0}^{\infty} \|\Psi_N\|^2_{\mathcal{H}_N}$. Thus we can

then identify $p_N := \|\Psi_N\|^2_{\mathcal{H}_N}$ as the probability

of finding the system with N particles with a wavevector $\frac{1}{\sqrt{p_N}} \Psi_N$ (which is a unit vector in $\mathcal{H}_N$).

* In principle, the spaces $\mathcal{H}_N$ need not to have anything to do with each other. However, the typical N-particle spaces have the following construction:

Standard constructions for $\mathcal{H}_N$

Suppose the system consists N similar particles, whose 1-particle space is $h =: \mathcal{H}_1$. (For instance, $h = L^2(\mathbb{R}^3)$ (spin-0 particle) or $h = \bigoplus_{\ell=1}^{2s+1} L^2(\mathbb{R}^3)$ (spin-s particle))

* The standard $\mathcal{H}_N$ is then defined as $\bigotimes_{n=1}^{N} h$.

$\Rightarrow$ For spin-0 particles $\mathcal{H}_N \cong L^2(\mathbb{R}^{3N})$.

* If there is no "physical observable" which can distinguish between the particles, the particles are called <u>indistinguishable</u> and it makes a lot of sense to "divide" out the particle-permutation symmetry from the beginning. The following examples are encountered in particle physics

  a) Bosons: wavevector is <u>symmetric</u> under permutation of particle labels
  b) Fermions: wavevector is <u>antisymmetric</u> ...

Case b) is possible, since only the probability densities $|\psi_N(x)|^2$ are thought to be observable properties, and these remain invariant under multiplications with $e^{i\varphi}, \varphi \in \mathbb{R}$, in particular, under $\underline{\psi} \to -\underline{\psi}$.

<u>An aside</u>: Permutation group $S_N :=$
$\{ \pi : \{1,2,\dots,N\} \circlearrowleft \mid \pi \text{ is bijective} \}$

Basic properties:
* $|S_N| = N!$
* <u>Transposition</u> (or swap) is a permutation which swaps two elements but leaves others invariant.
* Any $\pi \in S_N$ can be composed from finite number transpositions, and ...

... if the number is even, π has <u>even</u> parity and we define $\mathrm{sgn}(\pi) := +1$.
Otherwise, π has <u>odd parity</u>, and $\mathrm{sgn}(\pi) := -1$.
(These definitions make sense, since the evenness of the number of transposition depends only on π, not on the choice of transpositions used in the decomposition.)

* Commonplace (and convenient) notations:

$$(-1)^\pi := \mathrm{sgn}(\pi) \quad \text{and} \quad (+1)^\pi = +1 \;\forall \pi .$$

<u>Definition:</u> a) A vector $\Psi \in \bigotimes_{n=1}^{N} h$ is said to be <u>totally symmetric</u>, if $\forall \pi \in S_N$ and $\Phi \in \prod_{n=1}^{N} h$: $\left(\bigotimes_{n=1}^{N} \Phi_{\pi(n)}, \Psi\right) = \left(\bigotimes_{n=1}^{N} \Phi_n, \Psi\right)$

(i.e. if the corresponding multilinear map is ~~totally~~ symmetric)

b) Ψ is <u>totally antisymmetric</u> if $\forall \pi \in S_N$ and $\Phi \in \prod_{n=1}^{N} h$:

$$\left(\bigotimes_{n=1}^{N} \Phi_{\pi(n)}, \Psi\right) = (-1)^\pi \left(\bigotimes_{n=1}^{N} \Phi_n, \Psi\right)$$

($\Rightarrow$ sign-change under swaps)

<u>2.21.</u> Proposition

Denote $\mathcal{H}_N^{(+)} := \{ \Psi \in \bigotimes_{n=1}^{N} h \mid \Psi$ is ~~totally~~ symmetric$\}$
$\mathcal{H}_N^{(-)} := \{$ — " — $\mid$ — " — antisymmetric$\}$
Then both $\mathcal{H}_N^{(\sigma)}$, $\sigma = \pm 1$, are closed subspaces and the corresponding orthogonal projections $P_N^{(\sigma)}$ ...

... satisfy for any $\phi \in \prod_{k=1}^{N} h$ and either choice of the sign,

$$P_N^{(\pm)}\left(\bigotimes_{k=1}^{N} \phi_k\right) = \frac{1}{N!} \sum_{\pi \in S_N} (\pm 1)^{\pi} \bigotimes_{k=1}^{N} \phi_{\pi(k)}$$

Proof. Exercise 2.6. □

* Definition : a) Bosonic Fock space $= \mathcal{F}^{(+)} := \bigoplus_{N=0}^{\infty} \mathcal{H}_N^{(+)}$
($\mathcal{H}_0^{(+)} := \mathbb{C}$)
b) Fermionic Fock space $= \mathcal{F}^{(-)} := \bigoplus_{N=0}^{\infty} \mathcal{H}_N^{(-)}$
($\mathcal{H}_0^{(-)} := \mathbb{C}$)

* It might look unnecessarily complicated to work with the subspaces $\mathcal{F}^{(\pm)}$ instead of $\mathcal{F}$. However, the restriction to the subspace has some surprising consequences and simplifications. This is particularly so for antisymmetry, which can change the properties of an operator radically. (Stability of matter...)

2.22. An aside: Several species of particles.

If there are K different species of particles (Standard model of particle physics has 24 $s=\frac{1}{2}$ fermions (quarks and leptons & antiparticles) $1+3+8=12$ $s=1$ bosons (gauge bosons) and (usually) a Higgs boson with $s=0$. $\Rightarrow K=37.$), there are K possibly different 1-particle spaces $h^{(k)}$ and the Fock space is

$$\mathcal{F} = \bigoplus_{N \in \mathbb{N}_0^K} \mathcal{H}_N, \quad \text{with } \mathcal{H}_N = \bigotimes_{k=1}^{K} \mathcal{H}_{N_k}^{(k)}\ ; \quad \mathcal{H}_N^{(k)} = P_N^{(\sigma_k)}\left(\bigotimes_{n=1}^{N} h^{(k)}\right)$$

where $\sigma_k = -1$ if particle species k is fermionic and $\sigma_k = +1$ if it is bosonic.

* Of course, it is not known if this Fock space is the "right" space for the stand. model ($\exists$ dynamics?)

3. Bounded operators: $\mathcal{B}(\mathcal{H})$

* As in the general case in 2.2., we define for $V = \mathcal{H}$ the space

$$\mathcal{B}(\mathcal{H}) = \{ T : \mathcal{H} \to \mathcal{H} \mid T \text{ linear and } \|T\| < \infty \}$$

with $\|T\| = \sup \{ \|T\psi\| \mid \psi \in \mathcal{H}, \|\psi\| = 1 \}$.

* An <u>operator</u> on $\mathcal{H}$ is a <u>linear</u> mapping

$$A : D \to \mathcal{H}, \quad \text{with } D \subset \mathcal{H} \text{ subspace.}$$

$D = D(A)$ = <u>domain</u> of A.
$R(A) = \{ A\psi \mid \psi \in D \}$ = <u>range</u> of A.
$\mathrm{Ker}(A) = \{ \psi \in D \mid A\psi = 0 \}$ = <u>null space</u> or <u>kernel</u> of A

* $\mathcal{B}(\mathcal{H})$ is also called the set of <u>bounded operators</u>. Note that $T \in \mathcal{B}(\mathcal{H})$ implies $D(T) = \mathcal{H}$.

* $\mathcal{B}(\mathcal{H})$ = {the set of contiuous linear transformations of $\mathcal{H}$} (~~by Ex. 1.1~~)
Proof later.

* $\mathcal{B}(\mathcal{H})$ is a Banach space, since $\mathcal{H}$ is complete. (Proof later.)

* The following yields an important classification of bounded linear and sesquilinear functionals on $\mathcal{H}$:

<u>3.1. Thm:</u> a) Suppose $\Lambda : \mathcal{H} \to \mathbb{C}$ is <u>linear</u> and <u>bounded</u>. Then $\exists! \, \psi_0 \in \mathcal{H}$ s.t.

$$\Lambda\psi = (\psi_0, \psi) \quad \forall \psi \in \mathcal{H}.$$

b) Suppose $\Gamma : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is <u>bounded</u> and <u>sesquilinear</u>; that is, assume that

$\to$

(i) $\Gamma(\phi, \alpha\psi_1 + \beta\psi_2) = \alpha\Gamma(\phi,\psi_1) + \beta\Gamma(\phi,\psi_2)$,
$\Gamma(\alpha\phi_1 + \beta\phi_2, \psi) = \alpha^*\Gamma(\phi_1,\psi) + \beta^*\Gamma(\phi_2,\psi)$
$\forall\, \phi, \phi_1, \phi_2, \psi, \psi_1, \psi_2 \in \mathcal{H}$, $\alpha, \beta \in \mathbb{C}$.

(ii) $\exists\, C \geq 0$ s.t. $|\Gamma(\phi,\psi)| \leq C\|\phi\|\,\|\psi\|$ $\forall \phi, \psi \in \mathcal{H}$.

Then $\exists!\, T \in \mathcal{B}(\mathcal{H})$ s.t. $\forall \phi, \psi \in \mathcal{H}$

$$\Gamma(\phi,\psi) = (\phi, T\psi)$$

and $\|T\| = \sup\{ |\Gamma(\phi,\psi)| \mid \|\phi\| = 1 = \|\psi\| \} \leq C$.

pf: a) Boundedness of Λ means that (compare to **2.2.**)
$\|\Lambda\| := \sup\{ |\Lambda\psi| \mid \psi \in \mathcal{H}, \|\psi\| = 1\} < \infty$.
Let us start with uniqueness: If $\psi_0, \psi_0' \in \mathcal{H}$ s.t. $(\psi_0, \psi) = (\psi_0', \psi)$ $\forall \psi$ $\Rightarrow$
$(\psi_0 - \psi_0', \psi) = 0$ $\forall \psi$ $\Rightarrow$
$0 = (\psi_0 - \psi_0', \psi_0 - \psi_0') = \|\psi_0 - \psi_0'\|^2$
$\Rightarrow \psi_0' = \psi_0$. Thus ψ_0 is unique.
If $\Lambda = 0 \Rightarrow \Lambda\psi = 0 = (0,\psi)$ $\forall \psi \Rightarrow \psi_0 = 0$ is o.k. If $\Lambda \neq 0$, let $M = \mathrm{Ker}(\Lambda) := \{\psi \in \mathcal{H} \mid \Lambda\psi = 0\}$. Since Λ is bounded $\Rightarrow$ Λ continuous $\Rightarrow$ $M = \Lambda^{\leftarrow}(\{0\})$ is closed. It is also obviously a subspace. By Thm. 2.11 $\mathcal{H} = M \oplus M^\perp$. Now $M^\perp \neq \{0\}$ since else $M = \mathcal{H}$ which would mean $\Lambda = 0$. Thus $\exists\, \phi \in M^\perp$, $\phi \neq 0$.
However, then for any $\psi \in \mathcal{H}$
$\Lambda((\Lambda\psi)\phi - (\Lambda\phi)\psi) = (\Lambda\psi)(\Lambda\phi) - (\Lambda\phi)(\Lambda\psi) = 0$
$\Rightarrow (\Lambda\psi)\phi - (\Lambda\phi)\psi \in M$. Then by $\phi \in M^\perp$
$\Rightarrow (\Lambda\psi)\underbrace{(\phi,\phi)}_{\neq 0} - (\Lambda\phi)(\phi,\psi) = 0$

$$\Rightarrow \Lambda\psi = \frac{\Lambda\phi}{\|\phi\|^2}(\phi,\psi) = (\psi_0, \psi)$$

for $\psi_0 = \dfrac{(\Lambda\phi)^*}{\|\phi\|^2}\phi$. This proves a). ◢

b) Is a corollary of a): For any $\psi \in \mathcal{H}$ by (i),(ii) the map $\Lambda_\psi : \phi \longmapsto \Gamma(\phi,\psi)^*$ is linear, and $|\Lambda_\psi \phi| \leq C\|\psi\|$ if $\|\phi\| = 1$.

$\dashrightarrow$