~~2.8~~ 2.13. Defn. (internal direct sum in $\mathcal{H}$)

If $M_1, M_2 \subset \mathcal{H}$ are <u>closed</u> subspaces and $M_1 \cap M_2 = \{0\}$, we write

$$M_1 \oplus M_2 = \{ \psi_1 + \psi_2 \mid \psi_i \in M_i ; i=1,2 \} \subset \mathcal{H}.$$

$M_1 \oplus M_2$ is a closed subspace.
$\Rightarrow$ Further iteration is possible.

Example: Let $e_i \in \mathbb{C}^3$ be the unit vectors defined by $(e_i)_j = \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$ (= Kronecker delta.)
Let $M_i = \mathrm{span}(\{e_i\}) = \{ \alpha e_i \mid \alpha \in \mathbb{C} \}$
$\Rightarrow \mathbb{C}^3 = M_1 \oplus M_2 \oplus M_3$.

2.14. ~~2.9~~ Remark: For any $E \subset \mathcal{H}$, its orthogonal complement $E^\perp$ is a closed subspace.
(Linearity is obvious. It is closed as
$$E^\perp = \bigcap_{\phi \in E} \{\phi\}^\perp, \text{ and } \{\phi\}^\perp = f_\phi^{\leftarrow}(\{0\})$$
is closed, since $f_\phi : \mathcal{H} \to \mathbb{C}$ defined by $f_\phi(\psi) = (\phi, \psi)$ is continuous. (see 2.5.))

2.15. ~~2.10~~ <u>Defn:</u> If $M \subset \mathcal{H}$ is a <u>closed</u> subspace, and $\psi \in \mathcal{H}$, the set $E = \{ \psi - \phi \mid \phi \in M \}$ is non-empty, closed and convex. Therefore, by 2.7. $\exists! \, \phi_\psi \in M$ s.t $\|\psi - \phi_\psi\| \leq \|\psi - \phi\|$ $\forall \phi \in M$. Let $P : \mathcal{H} \to \mathcal{H}$ denote the mapping $\psi \mapsto \phi_\psi$. P is obviously a <u>projection</u> ($P^2 = P$) onto M.

Since $M^\perp$ is also a closed subspace, we can construct a projection Q onto M.

The following result proves that P and Q are, in fact, orthogonal projections.

2.16.
~~12.11~~ **Thm:** Let $M \subset \mathcal{H}$ be a closed subspace. Then

(i) $\mathcal{H} = M \oplus M^\perp$
(ii) The projections P onto M and Q onto $M^\perp$ are linear, and $P+Q=1$.
(iii) $\|\psi\|^2 = \|P\psi\|^2 + \|Q\psi\|^2 \quad \forall \psi \in \mathcal{H}$.

Pf: If $\psi \in M \cap M^\perp \Rightarrow \psi \in M$ and thus also $(\psi,\psi)=0 \Rightarrow \psi = 0$. Since $M, M^\perp$ are closed subspaces, $\exists\, M \oplus M^\perp \subset \mathcal{H}$.
Let $\psi_0 \in \mathcal{H}$ be arbitrary, and consider $\phi_0 = \psi_0 - P\psi_0$. By definition of P, then $\forall \phi \in M$: $\|\phi_0\| \le \|\psi_0 - \phi\|$
$= \|\psi_0 - P\psi_0 + P\psi_0 - \phi\| = \|\phi_0 + P\psi_0 - \phi\|$.
$\Rightarrow \forall \psi \in M: \|\phi_0\| \le \|\phi_0 + \psi\|$.
Thus if $\psi \in M$, also $\lambda\psi \in M \quad \forall \lambda \in \mathbb{C}$, and by 2.3.(iii) then $(\psi, \phi_0) = 0$.
Therefore, $\phi_0 \in M^\perp$ and $\psi_0 = P\psi_0 + \phi_0$, where $P\psi_0 \in M$. This proves (i).
In fact, $\phi_0 = Q\psi_0$: If $\phi \in M^\perp$, then

$$\|\psi_0 - \phi\|^2 = \|\underbrace{P\psi_0}_{\in M} + \underbrace{\phi_0 - \phi}_{\in M^\perp}\|^2 \underset{(P\psi_0,\, \phi_0-\phi)=0}{=} \|P\psi_0\|^2 + \|\phi_0 - \phi\|^2$$
$$\ge \|P\psi_0\|^2 = \|\psi_0 - \phi_0\|^2 .$$

Therefore, $\psi_0 = P\psi_0 + Q\psi_0$ and $1 = P+Q$.
Since $P\psi \perp Q\psi$, this implies also (iii).
Thus only the linearity in (ii) remains to be proven. For this, let $\psi, \phi \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$. By $1 = Q+P$, then

$$\psi = P\psi + Q\psi, \quad \phi = P\phi + Q\phi$$

and $\quad \alpha\psi + \beta\phi = P(\alpha\psi + \beta\phi) + Q(\alpha\psi + \beta\phi)$.
However, then also

$$\alpha P\psi + \beta P\phi + \alpha Q\psi + \beta Q\phi = \alpha\psi + \beta\phi$$

and thus

$$M \ni P(\alpha\psi + \beta\phi) - \alpha P\psi - \beta P\phi$$
$$= -Q(\alpha\psi + \beta\phi) + \alpha Q\psi + \beta Q\phi \in M^\perp$$

and $M \cap M^\perp = \{0\}$ implies

$$P(\alpha\psi + \beta\phi) = \alpha P\psi + \beta P\phi,$$
$$Q(\alpha\psi + \beta\phi) = \alpha Q\psi + \beta Q\phi. \qquad \square$$

2.7.

~~2.12~~ * If $E \subset \mathcal{H}$ is non-empty

$$\operatorname{span} E = \bigcup_{n=1} \left\{ \sum_{i=1}^{n} \alpha_i \psi_i \mid \alpha \in \mathbb{C}^n,\ \psi_i \in E\ \forall i \right\}.$$

* $E \subset \mathcal{H}$ is an <u>orthonormal set</u>, if a) $e', e \in E$, $e' \neq e \Rightarrow (e', e) = 0$
  b) $\|e\| = 1 \quad \forall e \in E$.

* A <u>maximal</u> orthonormal set in $\mathcal{H}$ is called an <u>orthonormal basis</u> (ONB) for the following reason:

<u>Thm.</u> Consider an orthonormal set $E = \{ e_i \mid i \in I \}$ in $\mathcal{H}$, indexed by an index set I. Then

E is maximal $\Leftrightarrow$ $\overline{\operatorname{span} E} = \mathcal{H}$.

If E is maximal,

a) $\forall \psi \in \mathcal{H}: \quad \|\psi\|^2 = \sum_{i \in I} |(e_i, \psi)|^2$

b) $\forall \phi, \psi \in \mathcal{H}: \quad (\phi, \psi) = \sum_{i \in I} (\phi, e_i)(e_i, \psi)$.

Pf: Rudin, RCA, 4.18. $\square$

* If $\mathcal{H}$ has a finite ONB, it is finite-dimensional.

* If $\mathcal{H}$ has a <u>countable</u> ONB, it is called <u>separable</u>.

* $L^2(\mathbb{R}^d)$ is separable for all $1 \le d < \infty$.

* There are also non-separable Hilbert spaces, for which I is uncountable. However, even then, for any $\psi \in \mathcal{H}$, the set $\{ i \in I \mid (e_i, \psi) \neq 0 \}$ is countable, and a) holds.

Direct sums and tensor products of Hilbert spaces

2.18. External direct sums

Let $\mathcal{H}_i$, $i \in I$, be a family of Hilbert spaces, where $I \neq \emptyset$ is some index set. Consider the following subset of the product space $\prod_{i\in I} \mathcal{H}_i$,

$$\mathcal{H} := \{ (\psi_i)_{i\in I} \mid \sum_{i\in I} \|\psi_i\|^2 < \infty \}$$

For $\Psi = (\psi_i)$ and $\Phi = (\phi_i)$ in $\mathcal{H}$, we define $\alpha\Psi$ and $\Psi + \Phi$ componentwise:

a) $(\alpha\Psi)_i := \alpha\psi_i \quad \forall i \in I, \ \alpha \in \mathbb{C}$

b) $(\Psi + \Phi)_i := \psi_i + \phi_i \quad \forall i$

Since $\|\alpha\psi_i\|^2 = |\alpha|^2\|\psi_i\|^2$ and $\|\psi_i + \phi_i\|^2 \leq 2(\|\psi_i\|^2 + \|\phi_i\|^2)$ (by Hölder's inequality) then $\alpha\Psi, \Psi+\Phi \in \mathcal{H}$. Also, then the set $I(\Psi) := \{ i \in I \mid \Psi_i \neq 0 \}$ is <u>countable</u> for all $\Psi \in \mathcal{H}$, and thus

$$\sum_{i\in I} \|\psi_i\|\,\|\phi_i\| = \sum_{i\in I(\Psi)\cup I(\Phi)} \|\psi_i\|\,\|\phi_i\| \overset{\text{Hölder}}{\leq} \sqrt{\sum_{i\in I(\Psi)} \|\psi_i\|^2 \sum_{i\in I(\Phi)} \|\phi_i\|^2}$$

is finite, and therefore

$$((\Psi, \Phi)) := \sum_{i\in I} (\psi_i, \phi_i)$$

defines a map $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$, by Cauchy-Schwarz.

<u>Theorem</u> $\mathcal{H}$ (with $((\cdot,\cdot))$) is a Hilbert space.

<u>Proof</u> Exercise 2.4. □

* <u>Notation</u>: then we write $\mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_i$.