2. Hilbert spaces

2.1. Definition

$\mathcal{H}$ is called a <u>Hilbert space</u> if it satisfies all of the follwing:

* $\mathcal{H}$ is a <u>complex</u> vector space.

* $\mathcal{H}$ has a <u>scalar product</u>:

  $(\cdot,\cdot) : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ s.t.

  a) $(\phi, \psi)^* = (\psi, \phi)$
  b) $(\phi, \psi_1 + \psi_2) = (\phi, \psi_1) + (\phi, \psi_2)$
  c) $(\phi, \alpha\psi) = \alpha(\phi, \psi)$
  $\qquad \forall \phi, \psi \in \mathcal{H}, \ \alpha \in \mathbb{C}$
  d) $(\psi, \psi) \geq 0 \quad \forall \psi \in \mathcal{H}$.
  e) $(\psi, \psi) = 0 \Rightarrow \psi = 0$.

* $\mathcal{H}$ is complete in the norm-topology given by

$$\|\psi\| := \sqrt{(\psi,\psi)} \quad \forall \psi \in \mathcal{H}. \tag{N}$$

Notes: • a) + b) + c) imply that $(\cdot,\cdot)$ is <u>sesquilinear</u>: it is linear in the second argument and conjugate-linear in the first argument.

2.2. Diversion: Norm and norm-topology.

Let V be a vector space. $\|\cdot\| : V \to \mathbb{R}$ is called a <u>norm</u> if

a) $\|\psi\| \geq 0 \quad \forall \psi \in V$
b) $\|\psi\| = 0 \Rightarrow \psi = 0$.
c) $\|\alpha\psi\| = |\alpha| \, \|\psi\| \quad \forall \alpha \in \mathbb{K}, \psi \in V$
d) $\|\psi + \phi\| \leq \|\psi\| + \|\phi\|. \quad \forall \psi, \phi \in V$.

Norm-topology is the topology defined using the metric

$$d(\psi, \phi) := \|\psi - \phi\|.$$

Continuity: Let X, Y be normed spaces. Then $F : X \to Y$ is continuous iff

$$\forall \psi \in X, \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.}$$
$$\|\phi - \psi\| < \delta \Rightarrow \|F(\phi) - F(\psi)\| < \varepsilon.$$

* If V is complete in the norm-metric, it is called a Banach space.

* theorem a) - e) $\Rightarrow$ $\|\cdot\|$ is a norm on $\mathcal{H}$.

Pf. d) $\Rightarrow$ $\|\psi\| \geq 0$ & e) $\Rightarrow$ $\|\psi\| = 0$ only if $\psi = 0$,
$\|\alpha\psi\|^2 = (\alpha\psi, \alpha\psi) = \alpha^* \alpha (\psi, \psi) = |\alpha|^2 \|\psi\|^2$.
the triangle inequality is proven in th. 2.3. □

* The set of bounded linear transformations of a normed space V is defined as

$$\mathcal{B}(V) := \{ \Lambda : V \to V \mid \Lambda \text{ linear and } \|\Lambda\| < \infty \}$$

where $\|\Lambda\| := \sup \{ \|\Lambda\psi\| \mid \psi \in V, \|\psi\| = 1 \}$.

- $\Lambda : V \to V$ linear is called bounded whenever $\|\Lambda\| < \infty$.
- $\mathcal{B}(V)$ is also a normed space with the above norm $\|\cdot\|$.

2.3. Theorem Assume a) - e). Then

$\forall \psi, \phi \in \mathcal{H}$:

(i) $|(\phi, \psi)| \leq \|\phi\| \|\psi\|$ (Cauchy-Schwarz)

(ii) $\|\phi + \psi\| \leq \|\phi\| + \|\psi\|$

(iii) $\|\psi\| \leq \|\psi + \lambda\phi\| \quad \forall \lambda \in \mathbb{C}$
$\Leftrightarrow (\phi, \psi) = 0$.

Proof: Let $\alpha = (\psi, \phi) \in \mathbb{C}$. Then $\forall \lambda \in \mathbb{C}$

$$
(*)\quad
\begin{aligned}
0 \le \|\psi + \lambda\phi\|^2 &= (\psi + \lambda\phi, \psi + \lambda\phi) \\
&= (\psi,\psi) + (\lambda\phi,\psi) + (\psi,\lambda\phi) + (\lambda\phi,\lambda\phi) \\
&= \|\psi\|^2 + |\lambda|^2\|\phi\|^2 + (\psi,\lambda\phi) + (\psi,\lambda\phi)^* \\
&= \|\psi\|^2 + |\lambda|^2\|\phi\|^2 + 2\,\mathrm{Re}(\lambda\alpha)
\end{aligned}
$$

If $\phi = 0 \Rightarrow (\psi,\phi) = 0 \quad \forall \psi \in \mathcal{H} \quad$ ((c))

$\Rightarrow \|\phi\|^2 = (\phi,\phi) = 0 \Rightarrow$ (i) holds.

If $\phi \neq 0$, choose $\lambda = -\frac{\alpha^*}{\|\phi\|^2}$

$$\Rightarrow 0 \le \|\psi\|^2 + \frac{|\alpha|^2}{\|\phi\|^4}\|\phi\|^2 - 2\frac{|\alpha|^2}{\|\phi\|^2}$$

$$= \|\psi\|^2 - \frac{|\alpha|^2}{\|\phi\|^2} \Rightarrow |\alpha|^2 \le \|\psi\|^2\|\phi\|^2$$

$\Rightarrow$ (i) holds,

Thus (i) has been proven. $\Rightarrow$

$$(\|\phi\| + \|\psi\|)^2 = \|\phi\|^2 + \|\psi\|^2 + 2\|\phi\|\,\|\psi\|$$
$$\underset{(i)}{\ge} \|\phi\|^2 + \|\psi\|^2 + 2|(\phi,\psi)|$$

But
$$
\begin{aligned}
\|\phi + \psi\|^2 &= (\phi+\psi, \phi+\psi) \\
&= \|\psi\|^2 + \|\phi\|^2 + 2\,\mathrm{Re}\,(\psi,\phi) \\
&\le \|\psi\|^2 + \|\phi\|^2 + 2\,|\mathrm{Re}(\psi,\phi)| \\
&\le \|\psi\|^2 + \|\phi\|^2 + 2\,|(\psi,\phi)|
\end{aligned}
$$

Thus (ii) holds, as well.

To prove (iii), note that if $\alpha = 0 \Rightarrow$ (by $(*)$)

$$\forall \lambda \in \mathbb{C}: \quad \|\psi + \lambda\phi\|^2 = \|\psi\|^2 + |\lambda|^2\|\phi\|^2$$
$$\ge \|\psi\|^2 \quad \Rightarrow \text{(iii) holds.}$$

If $\alpha \neq 0 \Rightarrow \phi \neq 0$ and thus by (i) and $(*)$

$$\|\psi + \lambda\phi\|^2 = \|\psi\|^2 - \frac{|\alpha|^2}{\|\phi\|^2} < \|\psi\|^2, \text{ for } \lambda = -\frac{\alpha^*}{\|\phi\|^2}.$$

$\Rightarrow$ (iii) does not hold. $\square$

2.4. Definitions: * From now on $\mathcal{H}$ denotes a Hilbert space.

* $\psi, \phi \in \mathcal{H}$ are orthogonal, denoted $\psi \perp \phi$, iff $(\psi,\phi) = 0$.

* If $E \subset \mathcal{H}$ any set, its orthogonal complement is defined as

$$E^\perp = \{\psi \in \mathcal{H} \mid (\phi,\psi) = 0 \ \forall \phi \in E\}$$

2.5. The mother of all Hilbert spaces:

Let μ be a positive measure on $\mathbb{X}$.
(For instance, $\mathbb{X} = \mathbb{R}^d$, $d\mu = dx$ = Lebesgue measure.)
Define

$$L^2_{pre}(\mu) = \left\{ \psi : \mathbb{X} \to \mathbb{C} \text{ measurable} \,\middle|\, \int_{\mathbb{X}} \mu(dx)\, |\psi(x)|^2 < \infty \right\}$$

and let

$$(\phi, \psi) = \int_{\mathbb{X}} \mu(dx)\, \phi(x)^* \psi(x) .$$

Then $(\cdot,\cdot)$ satisfies a) - d), but not e); Let $\psi' \sim \psi :\Leftrightarrow \psi' = \psi$ a.e.
$\Leftrightarrow \exists E \subset \mathbb{X}$, measurable, s.t. $\mu(E) = 0$ and $\forall x \notin E$: $\psi'(x) = \psi(x)$.
Then $\psi \sim 0 \Rightarrow (\psi, \psi) = 0$.
This, however, can be remedied easily:
"$\sim$" is an equivalence relation
$\Rightarrow$ can define $L^2(\mu) = L^2_{pre}(\mu) / \sim$
$=$ {set of equivalence classes w.r.t. $\sim$}
Since (ϕ, ψ) remains invariant if ϕ or ψ is modified on a set of measure zero, it yields a well-defined mapping
$L^2(\mu) \times L^2(\mu) \to \mathbb{C}$ (the integral is finite by Hölder's ineq.)
$L^2(\mu)$ is also a complex vector space, and

$$\sqrt{(\psi,\psi)} = \sqrt{\int_{\mathbb{X}} \mu(dx)\, |\psi(x)|^2} = L^p\text{-norm for } p = 2.$$

Thus by the completeness of L^p-spaces,(*) $L^2(\mu)$ is complete in $\|\cdot\| = \sqrt{(\cdot,\cdot)}$.

[$\therefore$ $L^2(\mu)$ is a Hilbert space.]

(*) See, for instance, Rudin: RCA, Th. 3.11.

* For μ = Lebesgue, we write $L^2(\mu) = L^2(\mathbb{R}^d)$.
* The following result (Rudin, RCA: Th. 3.12.) is useful about norm-convergent sequences:

Th. If $1 \le p \le \infty$ and $f_n \to f$ in $L^p(\mu)$-norm, then there is a subsequence $(n_k)_{k\in\mathbb{N}}$ s.t. $f_{n_k}(x) \xrightarrow{k\to\infty} f(x)$ a.e. $x \in X$.

Thus, for instance, any norm-convergent sequence in $L^2(\mu)$ has a pointwise a.e. convergent subseq.

Elementary properties of Hilbert spaces

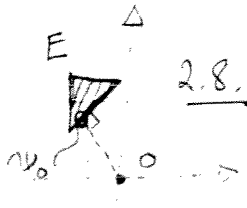
2.5. Thm: $\psi \mapsto \|\psi\|$ is a continuous mapping $\mathcal{H} \to \mathbb{R}$. $\forall \phi \in \mathcal{H}$ both $\psi \mapsto (\phi, \psi)$ and $\psi \mapsto (\psi, \phi)$ are continuous $\mathcal{H} \to \mathbb{C}$.
Pf: Chauchy-Schwarz (Rudin, RCA, Th. 4.6.) □

2.6. thm: If $M \subset \mathcal{H}$ is subspace, then its (norm-) closure $\overline{M}$ is a closed subspace. If M is also closed, it is a Hilbert space with the induced scalar product.
Pf: Rudin, RCA, Sect. 4.7. □

2.7. Definition: Let V be a vector space. $E \subset V$ is called convex, if

$$\phi, \psi \in E, \quad t \in [0,1] \Rightarrow (1-t)\psi + t\phi \in E.$$

that is, if the line connecting any $\phi, \psi \in E$ belongs to E.



2.8. Thm: If $E \subset \mathcal{H}$ is non-empty, closed, and convex, then $\exists!\, \psi_0 \in E$ s.t.
$\|\psi_0\| = \inf\{\|\psi\| \mid \psi \in E\}$
$\Leftrightarrow$ "ψ_0 is a norm-minimizer"

Proof: For any $\psi, \phi \in \mathcal{H}$ it holds that

$$\text{(PL)} \quad \|\psi + \phi\|^2 + \|\psi - \phi\|^2 = 2\|\psi\|^2 + 2\|\phi\|^2$$

(the "cross-terms" cancel; this identity is called the parallelogram law.)

Let $\delta := \inf\{\|\psi\| \mid \psi \in E\} \geq 0$. Consider $\psi, \phi \in E$, and apply (PL) to $\frac{1}{2}\psi$ and $\frac{1}{2}\phi$

$$\Rightarrow \frac{1}{4}\|\psi - \phi\|^2 = \frac{1}{2}\|\psi\|^2 + \frac{1}{2}\|\phi\|^2 - \left\|\frac{\psi+\phi}{2}\right\|^2$$

Since E is convex $\Rightarrow \frac{\psi+\phi}{2} \in E \Rightarrow \delta^2 \leq \left\|\frac{\psi+\phi}{2}\right\|^2$

Thus $\|\psi - \phi\|^2 \leq 2\|\psi\|^2 + 2\|\phi\|^2 - 4\delta^2$. $\forall \psi, \phi \in E$.
Therefore, if $\|\psi\| = \delta = \|\phi\| \Rightarrow \|\psi - \phi\|^2 \leq 0$
$\Rightarrow \psi = \phi$. This proves that any minimizer is unique.

For existence: By definition of δ, $\exists$ sequence $(\psi_n)_{n \in \mathbb{N}}$ s.t. $\psi_n \in E$ $\forall n$ and $\delta = \lim_{n\to\infty} \|\psi_n\|$.

Since then $\forall m, n$: $\|\psi_n - \psi_m\|^2$
$\leq 2\|\psi_n\|^2 + 2\|\psi_m\|^2 - 4\delta^2 \xrightarrow[n,m\to\infty]{} 0$,

(ψ_n) is a Cauchy sequence. As $\mathcal{H}$ is complete, $\exists\, \phi \in \mathcal{H}$ s.t. $\phi = \lim_{n\to\infty} \psi_n$, and in fact

then $\phi \in E$ since E is closed. $\|\cdot\|$ is continuous $\Rightarrow \delta = \lim_{n\to\infty} \|\psi_n\| = \|\lim \psi_n\| = \|\phi\|$.

Thus ϕ is a minimizer. $\square$

* Note that the proof relies heavily on (PL) which is a consequence of $\|\psi\|^2 = (\psi, \psi)$, i.e., of existence of scalar product.

(Abstract) completions

2.9. Recall the following topological construction:

If (V, d) is a metric space, then it can always be "completed": $\exists\ (\tilde{V}, \tilde{d})$ = complete metric space s.t. $V \subset \tilde{V}$, $\tilde{d}$-closure of $V = \tilde{V}$ and $\tilde{d}|_V = d$.

The construction is done as follows

Let $X := \{(x_n)_{n\in\mathbb{N}} \mid x_n \in V\ \forall n\ \&\ (x_n) \text{ is Cauchy}\}$
(= set of Cauchy sequences in V)
Define $(x_n) \sim (y_n)$ iff $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Then "$\sim$" is an equivalence relation on X and we define

$\tilde{V} := X/\sim$ and for $\tilde{x} = [(x_n)], \tilde{y} = [(y_n)] \in \tilde{V}$ we set $\tilde{d}(\tilde{x}, \tilde{y}) := \lim_{n\to\infty} d(x_n, y_n)$.

(If (x_n') and (y_n') are some other representatives of the classes, $|d(x_n', y_n') - d(x_n, y_n)|$
$\le |d(x_n', y_n') - d(x_n', y_n)| + |d(x_n', y_n) - d(x_n, y_n)|$
$\le d(y_n', y_n) + d(x_n', x_n) \to 0$.)
triangle inequality

We identify $x \in V$ with $[(x_n)]$, $x_n = x\ \forall n$, and then $\forall x, y \in V$: $\tilde{d}(x, y) = \lim_{n\to\infty} d(x, y) = d(x, y)$.

(If you are not familiar with the construction, prove that $(\tilde{V}, \tilde{d})$ has the stated properties as an exercise.)

* "Consistency check": If Y is a complete metric space and $V \subset Y$, then $\tilde{V} \cong \bar{V} \subset Y$.
(Proof. $y \in \bar{V} \Rightarrow \exists$ seq. $(x_n) \subset V$ s.t. $\lim_{n\to\infty} d(x_n, y) = 0$.
& (x_n) is Cauchy. Define $\Phi : \bar{V} \to \tilde{V}$ by setting ...

... $\Phi(y) = [(x_n)]$. If $\tilde{x} = [(x_n')] \in \tilde{V} \Rightarrow$
$(x_n') \subset V$ is d-Cauchy $\Rightarrow \exists\, y \in \overline{V}$ s.t. $\lim_{n\to\infty} x_n' = y$
But then $d(x_n, x_n') \le d(x_n, y) + d(y, x_n') \xrightarrow[n\to\infty]{} 0$
$\Rightarrow \tilde{x} = [(x_n)] = \Phi(y)$. Thus Φ is onto.
Consider any $y', y \in \overline{V}$ and the corresponding Cauchy sequences (x_n') and (x_n) in V. Then by the triangle inequality, $|d(y', y) - d(x_n', x_n)|$
$\le |d(y', y) - d(y', x_n)| + |d(y', x_n) - d(x_n', x_n)|$
$\le d(y, x_n) + d(y', x_n') \xrightarrow{n\to\infty} 0$
Thus $\tilde{d}(\Phi(y'), \Phi(y)) = \lim_{n\to\infty} d(x_n', x_n) = d(y', y)$
and Φ is an isometry. $\Rightarrow$ also 1-1
(since $\Phi(y') = \Phi(y) \Rightarrow \tilde{d}(\Phi(y'), \Phi(y)) = 0 \Rightarrow d(y', y) = 0$
$\Rightarrow y' = y$.) $\square$

<u>2.10</u> Suppose V is a vector space with scalar product $(\cdot,\cdot)$, and with the associated norm-topology: for $\psi, \phi \in V$ define $\|\psi\| = \sqrt{(\psi,\psi)}$ and $d(\psi,\phi) = \|\psi - \phi\|$.

By polarization identity (Exercise 2.2.), then $\forall \psi, \phi \in V$

$$(\phi, \psi) = \frac{1}{4}\Big(d(\phi, -\psi)^2 - d(\phi, \psi)^2 + i\, d(\phi, i\psi)^2 - i\, d(\phi, -i\psi)^2\Big)$$

where $\psi \pm \phi,\ \psi \pm i\phi \in V$. Using this, it is straightforward to prove that if we define

$$\forall\, \alpha \in \mathbb{C} \text{ and } \forall\, \tilde{\psi} = [(\psi_n)],\ \tilde{\phi} = [(\phi_n)] \in \tilde{V} = \mathfrak{X}/\sim$$

a) $\alpha\tilde{\psi} := [(\alpha\psi_n)]$
b) $\tilde{\psi} + \tilde{\phi} := [(\psi_n + \phi_n)]$
c) $((\tilde{\psi}, \tilde{\phi})) := \lim_{n\to\infty} (\psi_n, \phi_n)$

then $\tilde{V}$ is a Hilbert space with the scalar product $((\cdot,\cdot))$, and $V \subset \tilde{V}$ as before with $((\phi, \psi)) = (\phi, \psi)\ \forall \phi, \psi \in V$.

* $\tilde{V}$ is the completion of the scalar product space V.
It is also "consistent" as before: $V \subset \mathcal{H}$ = Hilbert sp.
$\Rightarrow \tilde{V} \cong \overline{V}$ and $((\cdot,\cdot)) = (\cdot,\cdot)_{\mathcal{H}}$.

2.11 For a normed space $\mathcal{V}$, the same construction yields a Banach space $\tilde{\mathcal{V}}$ when we define

c') $\|\tilde{\psi}\|_{\tilde{\mathcal{V}}} := \lim_{n \to \infty} \|\psi_n\|$.

2.12. Note that the following results are now immediate corollaries:

a) If $\mathcal{V} \subset \mathcal{B}$ = Banach space and $\mathcal{V}$ is a subspace, then $\overline{\mathcal{V}}$ = Banach space. (in the inherited norm)

b) If $\mathcal{V} \subset \mathcal{H}$ = Hilbert space and $\mathcal{V}$ is a subspace, then $\overline{\mathcal{V}}$ is a Hilbert space (in the inherited scalar product).