Appendices: (This material will not be part of the test.)

A) Proof of the Stone-Weierstrass theorem

* This will be a corollary of the Bishop's theorem which itself is a nice application of the functional analysis in topological vector spaces.

A.1. Definition: Suppose K is a compact non-empty Hausdorff space, and let $C(K)$ denote the C^*-algebra of continuous functions on $C(K)$. [Ex. 10.2.] Assume $\mathbb{A} \subset C(K)$ is a subalgebra, and for any $E \subset K$ define $\mathbb{A}|_E := \{ f|_E : E \to \mathbb{C} \mid f \in \mathbb{A} \}$. The set E is called $\mathbb{A}$-antisymmetric, if $g \in \mathbb{A}|_E$ with $g^* = g$ implies g is constant. (i.e. if every $f \in \mathbb{A}$ which is real on E is constant on E.) E is called maximal $\mathbb{A}$-antisymmetric if $E \subset E' = \mathbb{A}$-antisymm. $\Rightarrow E' = E$.

A.2. Proposition: Suppose K, $C(K)$ are as above, and $\mathbb{A} \subset C(K)$ is a subalgebra. Define $\lambda' \sim \lambda$ if $\exists E \subset K$ s.t. $\lambda', \lambda \in E$ and E is $\mathbb{A}$-antisymmetric. Then

a) "$\sim$" is an equivalence relation on K.

b) $E \subset K$ is maximal $\mathbb{A}$-antisymmetric $\iff \exists \lambda \in K$ s.t. $E = [\lambda]$

c) Every maximal $\mathbb{A}$-antisymmetric set is closed.

Proof: "a)" If $\lambda \in E$, then $\{\lambda\}$ is obviously $\mathbb{A}$-antisymmetric (any function on it is constant). Thus $\lambda \sim \lambda$.
If $\lambda' \sim \lambda$, obviously $\lambda \sim \lambda'$. If $\lambda' \sim \lambda_0$ and $\lambda'' \sim \lambda_0 \Rightarrow \exists E', E'' \subset K$ s.t. $\lambda_0, \lambda' \in E'$, $\lambda_0, \lambda'' \in E''$, and E', E'' are $\mathbb{A}$-antisymm.
$\Rightarrow \lambda_0, \lambda', \lambda'' \in E' \cup E'' =: E$ and if $g \in \mathbb{A}|_E$ with $g^* = g$ then $g' := g|_{E'}$ and $g'' := g|_{E''}$ satisfy $(g')^* = g'$ and $(g'')^* = g''$
$\Rightarrow g'$ and g'' are constant $\Rightarrow \forall \lambda \in E'$: $g(\lambda) = g'(\lambda) = g'(\lambda_0) = g(\lambda_0)$ and $\forall \lambda \in E''$: $g(\lambda) = g''(\lambda) = g''(\lambda_0) = g(\lambda_0) \Rightarrow \forall \lambda \in E: g(\lambda) = g(\lambda_0)$
$\Rightarrow g$ is constant. Thus E is $\mathbb{A}$-antisymmetric $\Rightarrow \lambda' \sim \lambda''$.
$\therefore$ "$\sim$" is an equivalence relation on K □

"b)" Suppose $\lambda_0 \in K$ and denote $E_0 := [\lambda_0]$. If $E' \subset K$ is s.t. $\lambda_0 \in E'$ and $E' \setminus E_0 \neq \emptyset$ $\Rightarrow \exists \lambda' \in E'$ s.t. $\lambda' \not\sim \lambda_0$
$\Rightarrow E'$ is not $\mathcal{A}$-antisymmetric. Thus if $g \in \mathcal{A}|_{E_0}$ with $g^* = g$, and $\lambda \in E_0$, then $\exists E \subset K$ s.t. $\lambda_0, \lambda \in E$ and E is $\mathcal{A}$-antisymm.
$\Rightarrow E \setminus E_0 = \emptyset \Rightarrow E \subset E_0 \Rightarrow (g|_E)^* = g|_E \Rightarrow g|_E$ is const.
$\Rightarrow g(\lambda) = g(\lambda_0)$. Thus g is const. $\therefore E_0$ is $\mathcal{A}$-antisymm. and if $E_0 \subset E' \subset K$ with E' $\mathcal{A}$-antisymm. $\Rightarrow E' \setminus E_0 = \emptyset \Rightarrow E' = E_0$.
Thus E_0 is maximal. Finally, if E is maximal $\mathcal{A}$-antisymm.
$\Rightarrow E \neq \emptyset$ (as $\exists p \in K \Rightarrow \emptyset \subsetneq \{p\}$) $\Rightarrow \exists \lambda_0 \in E \Rightarrow E \setminus [\lambda_0] = \emptyset$
$\Rightarrow E \subset [\lambda_0] = \mathcal{A}$-antisymm. $\overset{E \text{ max.}}{\Rightarrow} E = [\lambda_0]$. ■

"c)" E max. $\mathcal{A}$-antisymm. $\overset{\text{"b)"}}{\Rightarrow} E = [\lambda_0]$ for some $\lambda_0 \in K$. Set
$S_E := \{ f \in \mathcal{A} \mid (f|_E)^* = f|_E \}$.
Since $\mathcal{A}$ is a vector space $\Rightarrow 0 \in \mathcal{A} \Rightarrow 0 \in S_E \Rightarrow S_E \neq \emptyset$.
Denote $E' := \bigcap_{f \in S_E} f^{\leftarrow}(\{f(\lambda_0)\})$. As $\mathcal{A} \subset C(K)$, each $f \in S_E$ is continuous $\Rightarrow E'$ is closed in K and $\lambda_0 \in E'$. Also $f \in S_E \Rightarrow f|_E$ is const.
$\Rightarrow \forall \lambda \in E: f(\lambda) = f(\lambda_0) \Rightarrow \forall \lambda \in E: \lambda \in f^{\leftarrow}(\{f(\lambda_0)\}) \therefore E \subset E'$.
Finally, if $g \in \mathcal{A}|_{E'}$ with $g^* = g \Rightarrow \exists f \in \mathcal{A}$ s.t. $g = f|_{E'} \overset{E \subset E'}{\Rightarrow} f \in S_E$.
$\Rightarrow \forall \lambda \in E': f(\lambda) = f(\lambda_0) \Rightarrow g$ is const. Thus E' is $\mathcal{A}$-antisymm.
$\overset{E \text{ max.}}{\Rightarrow} E' = E \Rightarrow E$ is closed. □

A.3. Theorem (Bishop)

Suppose K is a compact non-empty Hausdorff space, and let $C(K)$ denote the corresponding C^*-algebra of continuous functions $K \to \mathbb{C}$. Assume that $\mathcal{A} \subset C(K)$ is closed and a subalgebra (but not necessarily a $*$-algebra) which contains all constant functions. If $g \in C(K)$ and $g|_E \in \mathcal{A}|_E$ for every maximal $\mathcal{A}$-antisymmetric set $E \subset K$, then $g \in \mathcal{A}$.

A.4. Corollary: Stone-Weierstrass theorem 18.10.' on p. 146.

Proof of 18.10.': By assumption, then $\mathcal{A}$ is a closed subalgebra. Suppose $\lambda, \lambda' \in K$, $\lambda' \neq \lambda \overset{\text{"d)"}}{\Rightarrow} \exists f \in \mathcal{A}$ s.t. $f(\lambda) \neq f(\lambda')$
$\Rightarrow$ either $\operatorname{Re} f(\lambda) \neq \operatorname{Re} f(\lambda')$ or $\operatorname{Im} f(\lambda) \neq \operatorname{Im} f(\lambda')$. Since both $\operatorname{Re} f = \frac{1}{2}(f + f^*) \in \mathcal{A}$ and $\operatorname{Im} f = \frac{1}{2i}(f - f^*) \in \mathcal{A}$ (by assumpt. e)), and $(\operatorname{Re} f)^* = \operatorname{Re} f$, $(\operatorname{Im} f)^* = \operatorname{Im} f$ we can conclude that $\lambda' \not\sim \lambda$.

Thus $[\lambda] = \{\lambda\}$ $\forall \lambda \in K$. If $g \in C(K)$, $\lambda_0 \in K$, the constant function $c: \lambda \mapsto g(\lambda_0)$ belongs to $\mathcal{A}$, and $g|_{[\lambda_0]} = c|_{[\lambda_0]}$.
$\therefore$ $g|_{[\lambda_0]} \in \mathcal{A}|_{[\lambda_0]}$ $\forall \lambda_0 \in K \overset{A.3.}{\Rightarrow} g \in \mathcal{A}$. $\therefore$ $\mathcal{A} = C(K)$ $\square$

Proof of A.3.: Suppose $\Lambda \in C(K)^* \overset{Ex. 8.2.}{\Rightarrow} \Lambda : C(K) \to \mathbb{C}$ is linear and $\forall f \in C(K)$ we have $|\Lambda(f)| \leq \|\Lambda\| \, \|f\|_\infty$
K is compact
$\overset{\dagger}{\Rightarrow}$ $\exists!$ regular complex Borel measure μ on K s.t.
$\Lambda(f) = \int_K \mu(d\lambda) f(\lambda)$ $\forall f \in C(K)$, [Riesz representation theorem, RCA, 6.19.]
Moreover, $\|\Lambda\| = |\mu|(K)$.
In addition, every such μ defines a $\Lambda_\mu \in C(K)^*$, since $|\int_K \mu(d\lambda) f(\lambda)| \leq \int_K |\mu|(d\lambda) \, \|f\|_\infty$ and $|\mu|(K) < \infty$.
Thus $C(K)^* = \mathcal{M}$ = regular complex Borel measures on K.

Define $\mathcal{A}^\perp := \{\mu \in \mathcal{M} \mid \int d\mu \, f = 0 \ \forall f \in \mathcal{A}\} \subset C(K)^*$, and $S := \{\mu \in \mathcal{A}^\perp \mid \|\mu\| \leq 1\}$ where $\|\mu\| := |\mu|(K)$. $(= \|\Lambda_\mu\|_{C(K)^*})$.
Since $\mathcal{A}^\perp = \bigcap_{f \in C(K)} \varphi_f^{\leftarrow}(\{0\})$, where $\varphi_f(\Lambda) := \Lambda(f)$ is weak*-continuous, $\mathcal{A}^\perp$ is weak*-closed. Now $S = \mathcal{A}^\perp \cap B^*$, where the unit ball B^* is weak*-compact (by Ex. 8.2.c) $\Rightarrow$ S is weak*-compact. If $\alpha \in \mathbb{K}$, with $|\alpha| \leq 1$, $t \in [0,1]$, $\mu_1, \mu_2 \in S$ and $f \in C(K)$, then $(\alpha\mu_1)(f) := \alpha\mu_1(f) = 0$, $\|\alpha\mu_1\| = |\alpha| \, \|\mu_1\| \leq 1$, $(t\mu_1 + (1-t)\mu_2)(f) = 0$, and $\|t\mu_1 + (1-t)\mu_2\| \leq t\|\mu_1\| + (1-t)\|\mu_2\| \leq t + 1 - t = 1$. Thus S is also balanced and convex.

Clearly, $0 \in S$, and if $S = \{0\}$, then $\mathcal{A}^\perp = \{0\}$ (as $\mu \in \mathcal{A}^\perp$, $\mu \neq 0 \Rightarrow \frac{1}{\|\mu\|}\mu \in \mathcal{A}^\perp$ and $\|\frac{1}{\|\mu\|}\mu\| = 1 \Rightarrow 0 \neq \frac{1}{\|\mu\|}\mu \in S$.). By assumption, $\mathcal{A}$ is a closed subspace of $C(K)$. Thus, if $f_0 \in C(K) \setminus \mathcal{A}$, then $\exists \mu \in \mathcal{M}$ s.t. $\mu(f_0) = 1$ and $\mu(f) = 0$ $\forall f \in \mathcal{A}$ (Hahn-Banach 12.7.b) $\Rightarrow \mu \in \mathcal{A}^\perp \Rightarrow \mu = 0$ $\lightning$. Therefore, then $\mathcal{A} = C(K)$, and the theorem trivially holds.

Assume thus $S \neq \{0\}$. Let μ_0 be an extreme point of S. Then $\mu_0 \neq 0$, since $\exists \mu_1 \in S$ s.t. $\mu_1 \neq 0 \overset{S \text{ bal.}}{\Rightarrow} -\mu_1 \in S$ and $0 = \frac{1}{2}\mu_1 + \frac{1}{2}(-\mu_1)$, where $\mu_1 \neq -\mu_1$. But then $\|\mu_0\| = 1$, since else $0 < \|\mu_0\| < 1$ and $\mu_0 = \|\mu_0\| \underbrace{\frac{1}{\|\mu_0\|}\mu_0}_{\|\cdot\| = 1 \Rightarrow \in S} + (1 - \|\mu_0\|) \underbrace{0}_{\in S}$.
Define $E := \bigcap \mathcal{P}$, with $\mathcal{P} := \{C \subset K \mid C \text{ compact and } |\mu_0|(C) = 1\}$. Since K is compact Hausdorff, C compact $\Rightarrow$ C closed $\Rightarrow$ E closed $\Rightarrow$ E compact.
By Riesz, $|\mu_0|$ is a regular Borel measure with $|\mu_0|(K) = 1$, and thus $|\mu_0|(E) = \inf\{\mu(V) \mid E \subset V, V \text{ open in } K\}$. However, if V is open and $E \subset V \Rightarrow V^c \subset E^c = \bigcup_{C \in \mathcal{P}} C^c$. As V^c is closed $\Rightarrow$ compact, and

each C^c is open $\Rightarrow \exists F \subset \mathcal{P}$, with $|F|<\infty$, s.t. $V^c \subset \bigcup_{C\in F} C^c$
$\Rightarrow |\mu_0|(V^c) \le \sum_{C\in F} |\mu_0|(C^c) = 0$ since $\forall C\in \mathcal{P}$, $|\mu_0|(C) = 1 = |\mu_0|(K)$
$= |\mu_0|(C) + |\mu_0|(C^c) \Rightarrow |\mu_0|(C^c) = 0$.
Thus $|\mu_0|(V) = 1 - |\mu_0|(V^c) = 1$. Therefore, $|\mu_0|(E) = 1$.
$\Rightarrow \forall f$ measurable: $\int_K \mu_0(d\lambda) f(\lambda) = \int_E \mu_0(d\lambda) f(\lambda)$. ($E$ is called the support of μ_0.)

Suppose then that $f_0 \in \mathcal{A}$ is s.t. $0 < f_0(\lambda) < 1$ $\forall \lambda \in E$. Define $\mu_1(d\lambda) = f_0(\lambda)\mu_0(d\lambda)$ and $\mu_2(d\lambda) = (1-f_0(\lambda))\mu_0(d\lambda)$ $\Rightarrow \mu_1, \mu_2 \in M$ and $\forall f \in \mathcal{A}$: $\mu_1(f) = \mu_0(f f_0) = 0$ and $\mu_2(f) = \mu_0((1-f_0)f) = 0$ since $1 \in \mathcal{A} \Rightarrow f(1-f_0), f f_0 \in \mathcal{A}$. Thus $\mu_1, \mu_2 \in \mathcal{A}^\perp$. Since $f_0 > 0$
$\Rightarrow \exists \varepsilon > 0$ s.t. $|\mu_0|(F_\varepsilon) > 0$, $F_\varepsilon := \{\lambda \in E \mid f_0(\lambda) \ge \varepsilon\}$
(else $|\mu_0|(E) = |\mu_0|(\bigcup_{n\in\mathbb{N}_+} F_{\frac{1}{n}}) = 0$ $\lightning$)
$\Rightarrow |\mu_1|(K) = \int_K |\mu_0|(d\lambda) f_0(\lambda) = \int_E |\mu_0|(d\lambda) f_0(\lambda) \ge \int_{F_\varepsilon} |\mu_0|(d\lambda) f_0(\lambda)$
$\ge \varepsilon |\mu_0|(F_\varepsilon) > 0$. Similarly, $|\mu_2|(K) > 0$. Thus $\|\mu_1\|, \|\mu_2\| > 0$.
In addition,

$$\|\mu_1\| + \|\mu_2\| = \int_K |\mu_0|(d\lambda)\, |f_0(\lambda)| + \int_K |\mu_0|(d\lambda)\, |1 - f_0(\lambda)|$$
$$= \int_E |\mu_0|(d\lambda)\,(f_0(\lambda) + 1 - f_0(\lambda)) = |\mu_0|(E) = 1.$$

$$\Rightarrow \mu_0 = \mu_1 + \mu_2 = \|\mu_1\| \underbrace{\frac{1}{\|\mu_1\|}\mu_1}_{\in S} + (1 - \|\mu_1\|) \underbrace{\frac{1}{\|\mu_2\|}\mu_2}_{\in S}$$

Since μ_0 is an extreme point of S $\Rightarrow \mu_1 = \|\mu_1\|\mu_0$ and $\mu_2 = \|\mu_2\|\mu_0$
$\Rightarrow 0 = \|\mu_1 - \|\mu_1\|\mu_0\| = \|(f_0 - \|\mu_1\|)\mu_0\| = \int_K |\mu_0|(d\lambda)\, |f_0(\lambda) - \|\mu_1\||$.
$(= \int_E |\mu_0|(d\lambda)\, |f_0(\lambda) - \|\mu_1\||)$ Set $\tilde{F} := \{\lambda \in K \mid f_0(\lambda) = \|\mu_1\|\}$.
Since f_0 is contin. $\Rightarrow f_0 - \|\mu_1\|$ is contin. $\Rightarrow \tilde{F}$ is closed
$\Rightarrow \tilde{F}$ is compact. Also $|\mu_0|(\tilde{F}^c) = 0$, as $\tilde{F}^c = \bigcup_{n\in\mathbb{N}_+} \{\lambda \in K \mid |f_0(\lambda) - \|\mu_1\|| > \frac{1}{n}\}$,
and $\int_K |\mu_0|(d\lambda)\, \mathbb{1}(|f_0(\lambda) - \|\mu_1\|| > \frac{1}{n})$
$\le n \int_K |\mu_0|(d\lambda)\, |f_0(\lambda) - \|\mu_1\|| = 0$. $\forall n \in \mathbb{N}_+$.
Thus $|\mu_0|(\tilde{F}) = 1 \Rightarrow E \subset \tilde{F} \Rightarrow f_0(\lambda) = \|\mu_1\|$ $\forall \lambda \in E$. Therefore, if $f \in \mathcal{A}$ and $f(\lambda) \in \mathbb{R}$ $\forall \lambda \in E \overset{E \text{ compact}}{\Rightarrow} f(E)$ is compact, and $\exists m, M \in \mathbb{R}$ s.t. $m \le f(\lambda) < M$ $\forall \lambda \in E$. Set $f_0(\lambda) = \frac{f(\lambda) - m}{2(M-m)} + \frac{1}{4}$ $\forall \lambda \in K$.
Since $\mathcal{A}$ contains constants $\Rightarrow f_0 \in \mathcal{A}$. But as $\frac{1}{4} \le f_0(\lambda) \le \frac{3}{4}$, we can conclude that $\exists c_0 \in (0,1)$ s.t. $f_0|_E = c_0$
$\Rightarrow f(\lambda) = m + 2(M-m)(c_0 - \frac{1}{4})$ $\forall \lambda \in E$ $\Rightarrow f|_E$ is const.
$\therefore$ $E = \operatorname{supp} \mu_0$ is $\mathcal{A}$-antisymmetric if μ_0 is an extreme point of S.

Suppose then $g \in C(K)$ is such that $g|_M \in \mathcal{A}|_M$ $\forall M =$ max. $\mathcal{A}$-antis.
Let $S_0 = \{\mu_0 \mid \mu_0$ is an extreme point of $S\}$. Since S is compact and convex, by Krein-Milman (14.2.) $S = \overline{\text{Hull}(S_0)}$, closure in weak*-topology. If $\mu_0 \in S_0$, $\Rightarrow E_0 := \operatorname{supp} \mu_0$ is $\mathcal{A}$-antisymm. $\Rightarrow \exists M =$ max. $\mathcal{A}$-antis. s.t. $E_0 \subset M \Rightarrow \exists f \in \mathcal{A}$ s.t. $f|_M = g|_M \Rightarrow g|_{E_0} = f|_{E_0}$

$$\Rightarrow \int_K d\mu_0\, g = \int_{E_0} d\mu_0\, g = \int_{E_0} d\mu_0\, f \overset{\mu_0 \in S}{=} 0.$$

Thus $\int_K d\mu\, g = 0 \quad \forall k \in \mathrm{Hull}(S_0)$; (by linearity).
Since the map $\varphi_g : \mu \mapsto \mu(g)$ is weak*-continuous,
$\Rightarrow S = \overline{\mathrm{Hull}(S_0)} \subset \varphi_g^{\leftarrow}(\{0\})$. Thus $\forall \mu \in S : \int_K d\mu\, g = 0$
$\Rightarrow \forall \mu \in A^\perp : \int_K d\mu\, g = 0$. However, if $g \in C(K) \setminus A$ $\exists \mu \in M$ s.t. $\mu(g) = \int_K d\mu\, g = 1$ and $\mu \in A^\perp$. Therefore, now $g \in A$ □

* [Rudin, FA, 5.8] has an interesting application of Bishop's theorem. Roughly, it states that if a collection of holomorphic function are parameterized continuously on a compact subset of $\mathbb{R}^N$, $N \in \mathbb{N}_+$, then the functions can be uniformly approximated by polynomials <u>also in the parameters</u>. Note that compact sets in $\mathbb{R}^N$, $N \geq 2$, can be quite complicated: unions of infinitely many points, lines, etc.

B. Proof of Fuglede–Putnam–Rosenblum Theorem

The proof relies on the exponential of an operator, defined as

$$\exp(T) := \sum_{n=0}^{\infty} \frac{1}{n!} T^n \qquad \forall T \in \mathcal{B}(\mathcal{H}).$$

B.1. Proposition: Suppose $T, S \in \mathcal{B}(\mathcal{H})$.

a) $\exp(T) = \text{norm-}\lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{n!} T^n$ and $\|\exp(T)\| \leq e^{\|T\|}$.

b) $(\exp(T))^* = \exp(T^*)$

c) If $TS = ST$, then $\exp(T+S) = \exp(T)\exp(S) = \exp(S)\exp(T)$.

d) $\exp(T)$ is invertible, and $\exp(T)^{-1} = \exp(-T)$.

e) The map $\lambda \mapsto \exp(\lambda T)$ defines an entire function on $\mathbb{C}$, whose derivative is $T\exp(\lambda T) = \exp(\lambda T)T$.

Proof: "a) & c)" are proven exactly as Ex. 1.3. & Ex. 1.4.
"b)" follows from continuity of "*" and a).
"d)" As $[T, -T] = 0 \overset{c)}{\Rightarrow} \exp(T)\exp(-T) = \exp(-T)\exp(T)$
$= \exp(T-T) = \exp(0) = \mathbb{1}$. $\Rightarrow \exp(-T) = \exp(T)^{-1}$.
"e)" By "a)" and 16.2., $T\exp(\lambda T) = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{n!} T^{n+1} = \exp(\lambda T)T$.

Similarly, $\forall \lambda \in \mathbb{C}$: $\exp(\lambda T) - 1 - \lambda T$

$$= \lim_{N\to\infty}\left(\sum_{n=0}^{N} \frac{1}{n!}(\lambda T)^n - 1 - \lambda T\right) = \lim_{N\to\infty} \sum_{n=2}^{N} \frac{1}{n!}(\lambda T)^n$$

$$\Rightarrow \|\exp(\lambda T) - 1 - \lambda T\| \le \sum_{n=2}^{\infty} \frac{1}{n!} |\lambda|^n \|T\|^n \le |\lambda|^2 \|T\|^2 e^{|\lambda|\,\|T\|}$$

$$\Rightarrow \forall \lambda, z \in \mathbb{C} : \|\exp((\lambda+z)T) - \exp(zT) - \lambda T \exp(zT)\|$$

$$\overset{c)}{=} \|\exp(zT)(\exp(\lambda T) - 1 - \lambda T)\| \le e^{|z|\,\|T\|} |\lambda|^2 \|T\|^2 e^{|\lambda|\,\|T\|}$$

$$\Rightarrow \frac{1}{\lambda}(\exp((\lambda+z)T) - \exp(zT)) \to T\exp(zT) \text{ as } \lambda \to 0.$$

$\Rightarrow$ strongly holom. on $\mathbb{C}$. $\square$

<u>B.2. Lemma</u>: If $U \in \mathcal{B}(\mathcal{H})$ is unitary, and $\mathcal{H} \neq \{0\}$, then $\|U\| = 1$.

<u>Proof</u>: $\forall \psi \in \mathcal{H} : \|U\psi\|^2 = (U\psi, U\psi) = (\psi, U^*U\psi)$
$= (\psi, \psi) = \|\psi\|^2 \Rightarrow \|U\| = 1$, if $\exists \psi \in \mathcal{H}, \psi \neq 0$ $\square$

<u>Proof of 18.10.''</u>: The assumptions are $M, N, T \in \mathcal{B}(\mathcal{H})$ and
Assume thus $\mathcal{H} \neq \{0\}$. $MM^* = M^*M$, $NN^* = N^*N$, and $MT = TN$. (*)
Now for any $S \in \mathcal{B}(\mathcal{H})$, set $V = S - S^* \Rightarrow V^* = -V$
$\Rightarrow \exp(V)^* = \exp(-V) = \exp(V)^{-1} \Rightarrow \exp(V)$ is unitary
$\therefore \|\exp(S - S^*)\| = 1 \ \forall S \in \mathcal{B}(\mathcal{H})$.

By induction, we find $M^nT = TN^n \ \forall n \in \mathbb{N}_+$.
$(M^{n+1}T = M^n(MT) = M^nTN \overset{\text{ind. ass.}}{=} TN^nN = TN^{n+1}.)$ Thus

$$\exp(M)T = \lim_{N\to\infty}\left(\sum_{n=1}^{N} \frac{1}{n!} M^nT + T\right) = \lim_{N\to\infty}\left(\sum_{n=1}^{N} \frac{1}{n!} TN^n + T\right) = T\exp(N).$$

$\overset{B.1.d)}{\Rightarrow} T = \exp(-M)T\exp(N)$. Since $[M, M^*] = 0$ and $[N, N^*] = 0$

$$\overset{B.1.c)}{\Rightarrow} \exp(M^*)T\exp(-N^*) = \exp(\underbrace{M^* - M}_{= M^* - (M^*)^*})T\exp(N - N^*)$$

$\Rightarrow \|\exp(M^*)T\exp(-N^*)\| \le \|T\|$.

For $\lambda \in \mathbb{C}$, define $f(\lambda) := \exp(\lambda M^*)T\exp(-\lambda N^*)$
By B.1.e), then f is entire and
$f'(\lambda) = \exp(\lambda M^*)(M^*T - TN^*)\exp(-\lambda N^*)$. However, as
λ^*M and λ^*N are also normal and $(\lambda^*M)T = \lambda^*(MT) = T(\lambda^*N)$,
$\Rightarrow \|f(\lambda)\| = \|\exp((\lambda^*M)^*)T\exp(-(\lambda^*N)^*)\| \le \|T\| \ \forall \lambda \in \mathbb{C}$.
$\Rightarrow f$ is bounded and entire $\overset{15.7.}{\Rightarrow} f$ is constant
$\Rightarrow f' = 0 \Rightarrow f'(0) = 0 = M^*T - TN^* \Rightarrow M^*T = TN^*$ $\square$

(*) If $\mathcal{H} = \{0\} \Rightarrow \mathcal{B}(\mathcal{H}) = \{0\} \Rightarrow M^*T = 0 = TN^*$.

C. Holomorphic functions of operators

C.1. Definition: Suppose $\mathcal{A}$ is a Banach algebra, $\Omega \subset \mathbb{C}$ is open and nonempty, and $f : \Omega \to \mathbb{C}$ is holomorphic. Denote $\mathcal{A}_\Omega := \{x \in \mathcal{A} \mid \sigma(x) \subset \Omega\}$, and define as a vector valued integral

$$\tilde{f}(x) := \frac{1}{2\pi i} \oint_{\Gamma_x} dz\, f(z)\, R_z(x) \qquad \forall x \in \mathcal{A}_\Omega$$

where Γ_x is any path (cycle) in Ω which surrounds $\sigma(x)$, i.e., such that $\operatorname{Ind}_{\Gamma_x}(z) = 1$ for $z \in \sigma(x)$, $\operatorname{Ind}_{\Gamma_x}(z) = 0$ if $z \notin \Omega$, and $\Gamma_x(t) \in \Omega \setminus \sigma(x)$ $\forall t$.

* By 16.14. $z \mapsto R_z(x)$ is holomorphic in $\Omega \setminus \sigma(x)$ and thus by 15.8. $\tilde{f}(x) \in \mathcal{A}$, and the value does not depend on the choice of Γ_x.

* If $x \in \mathcal{B}(\mathcal{H})$ is normal, symbolic calculus defines $f(x)$ for all Borel functions. If $f \in H(\Omega)$ and $\sigma(x) \subset \Omega$, these two definitions agree: $\forall \psi, \varphi \in \mathcal{H}$

$$(\varphi, \tilde{f}(x)\psi) = \frac{1}{2\pi i} \oint_{\Gamma_x} dz\, f(z)\, (\varphi, R_z(x)\psi)$$

$$\overset{19.11.\lambda)}{=} \frac{1}{2\pi i} \oint_{\Gamma_x} dz\, f(z) \left[\int_{\sigma(x)} E_{\varphi,\psi}(d\lambda)\, (z-\lambda)^{-1} \right]$$

$$\overset{\text{Fubini}}{=} \int_{\sigma(x)} E_{\varphi,\psi}(d\lambda) \left[\frac{1}{2\pi i} \oint_{\Gamma_x} dz\, f(z)(z-\lambda)^{-1} \right]$$

$\lambda \in \sigma(x)$, and $\operatorname{Ind}_{\Gamma_x}(\lambda) = 1$.

$$\overset{\text{Cauchy}}{=} \int_{\sigma(x)} E_{\varphi,\psi}(d\lambda)\, f(\lambda) = (\varphi, f(x)\psi).$$

$$\Rightarrow \tilde{f}(x) = f(x).$$

* In general, $\tilde{f}(x)$ satisfies properties similar to 19.11., see [Rudin, FA, 10.22. – 10.44]. It has the benefit that x does not need to be normal, but the drawback is that $f : \Omega \to \mathbb{C}$ has to be analytic.

* Note also that for exp defined in Appendix B, one also has $\exp(T) = \tilde{f}(T)$, with $f(\lambda) := e^\lambda$ $\forall \lambda \in \mathbb{C}$, and for any $T \in \mathcal{B}(\mathcal{H})$.

Proof. Suppose $T \in \mathcal{B}(\mathcal{H})$.

By B.1. a) $\Rightarrow \forall \varphi, \psi \in \mathcal{H}$

$$(\varphi, \exp(T)\psi) = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{n!} (\varphi, T^n \psi) = \sum_{n=0}^{\infty} \frac{1}{n!} (\varphi, T^n \psi)$$

On the other hand, $(\varphi, \tilde{f}(T)\psi) = \frac{1}{2\pi i} \oint_{\Gamma_T} dz\, e^z (\varphi, R_z(T)\psi)$

where $|(\varphi, R_z(T)\psi)| \le \frac{1}{\varepsilon} \|\varphi\| \|\psi\|$ $\forall z \in \rho(\mathcal{E}_x)$ for some $\varepsilon > 0$.

$\Rightarrow$ By Fubini

$$(\varphi, \tilde{f}(T)\psi) = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\frac{1}{2\pi i} \oint_{\Gamma_T} dz\, z^n (\varphi, R_z(T)\psi)}$$

$$= (\varphi, \Big(\frac{1}{2\pi i} \oint_{\Gamma_T} dz\, z^n R_z(T)\Big)\psi)$$

By Cauchy, and Ex. 11.4., here $\frac{1}{2\pi i} \oint_{\Gamma_T} dz\, R_z(T) = 1$.

Then by induction, $\frac{1}{2\pi i} \oint_{\Gamma_T} dz\, z^n R_z(T) = T^n \;\forall n \in \mathbb{N}$,

as was done in Ex. 11.4.: $z R_z(T) = (z1 - T + T)(z1 - T)^{-1}$
$= 1 + T(z1-T)^{-1} = 1 + TR_z(T) \Rightarrow \frac{1}{2\pi i} \oint_{\Gamma_T} dz\, z^{n+1} R_z(T)$

$$= \frac{1}{2\pi i} \oint_{\Gamma_T} dz \left(z^n 1 + T z^n R_z(T)\right) = 0 + T \frac{1}{2\pi i} \oint_{\Gamma_T} dz\, z^n R_z(T) = T^{n+1},$$

using the induction assumption. Thus $(\varphi, \tilde{f}(T)\psi) = \sum_{n=0}^{\infty} \frac{1}{n!} (\varphi, T^n \psi)$
$= (\varphi, \exp(T)\psi)$ $\forall \varphi, \psi$
$\Rightarrow \exp(T) = \tilde{f}(T)$ $\square$

D. Other formulations of the spectral decomposition theorem

* See Reed & Simon I, chapter VII.