**18.12. Definition:** Suppose $\mathbb{A}$ is a Banach algebra with involution. $x \in \mathbb{A}$ is called *positive*, denoted $x \geq 0$, if $x^* = x$ and $\sigma(x) \subset [0,\infty)$.

**18.13. Theorem:** Suppose $\mathbb{A}$ is a $C^*$-algebra, and $x, y \in \mathbb{A}$. Then

a) $x^* = x \Rightarrow \sigma(x) \subset \mathbb{R}$
b) $x$ normal $\Rightarrow r_\sigma(x) = \|x\|$
c) always $r_\sigma(xx^*) = r_\sigma(x^*x) = \|x\|^2$
d) $x \geq 0, y \geq 0 \Rightarrow x+y \geq 0$
e) always $xx^* \geq 0$ and $x^*x \geq 0$
f) $e + x^*x,\ e + xx^* \in G(\mathbb{A})$

Proof: "a)" Assume $x_0^* = x_0$ and define $E := \{x_0\}$.
$\Rightarrow E$ is normal $\overset{18.7.a)}{\Rightarrow} \exists M$ which maximally normal and $E \subset M$. $\overset{18.7.b)}{\Rightarrow} M$ is a commutative Banach algebra, and as $x_0 \in M$, $\sigma_M(x_0) = \sigma_{\mathbb{A}}(x_0)$. As $M$ is normal, $*|_M$ is an involution on $M$ which makes it into a commutative $C^*$-algebra. $\overset{18.10.}{\Rightarrow}$ The Gelfand transform $g: x \mapsto \hat{x}$ is an isometric $*$-isomorphism $M \to \Delta_M$ = maximal ideal space of $M$. Also, as $x_0$ is self-adjoint, $\hat{x}_0 \in \Delta_M$ is real-valued, and by 17.8.c), $\hat{x}_0(\Delta_M) = \sigma_M(x_0) = \sigma_{\mathbb{A}}(x_0) \Rightarrow \sigma(x_0) := \sigma_{\mathbb{A}}(x_0) \subset \mathbb{R}$. ∎

"b)" If $x_0$ is normal $\Rightarrow E := \{x_0, x_0^*\}$ is normal $\overset{18.7.}{\Rightarrow} \exists M, \Delta_M, g$ as in "a)" and $\sigma_M(x_0) = \sigma(x_0)$. Since $g$ is an isometry,

$$r_\sigma(x) := \sup_{\lambda \in \sigma(x)} |\lambda| = \sup_{h \in \Delta_M} |\hat{x}_0(h)| = \|\hat{x}_0\|_\infty = \|g(x_0)\|_\infty = \|x_0\|. \ ∎$$

"c)" By 18.4. a), $xx^*$ and $x^*x$ are self-adjoint $\Rightarrow$ normal. Then by "b)" $\Rightarrow r_\sigma(xx^*) = \|xx^*\| \overset{18.5.}{=} \|x\|^2 = \|x^*x\| = r_\sigma(x^*x)$ ∎

"d)" Assume $x', x \geq 0$. Let $r := \|x\|$, $r' := \|x'\|$, $u := x' + x$, $s = r + r'$. By assumption, $\sigma(x) \subset [0,\infty)$ and $x^* = x$ $\Rightarrow x$ is normal $\overset{b)}{\Rightarrow} \sigma(x) \subset [0, r]$. Similarly, $\sigma(x') \subset [0, r']$. Since $x \in G(\mathbb{A}) \Rightarrow -x \in G(\mathbb{A})$ ( $(-x)^{-1} = -x^{-1}$ ) we have $\lambda \in \rho(re - x) \Leftrightarrow \lambda e - (re - x) = -(re - \lambda e - x) \in G(\mathbb{A})$
$\Leftrightarrow (r-\lambda)e - x \in G(\mathbb{A}) \Leftrightarrow r - \lambda \in \rho(x)$. Thus $\sigma(re - x) = r - \sigma(x) \subset [0, r]$.
$(re-x)^* = re-x$ $\overset{b)}{\Rightarrow} \|re - x\| = r_\sigma(re - x) \leq r$. Similarly, $\|r'e - x'\| \leq r'$.
Therefore, $\|se - u\| \leq \|re - x\| + \|r'e - x'\| \leq r + r' = s$.
Since $(se-u)^* = se^* - (x'+x)^* = se - u \overset{a),b)}{\Rightarrow} \sigma(se-u) \subset \mathbb{R}$, $r_\sigma(se-u) \leq s$

$\Rightarrow \sigma(se-u) \subset [-s, s]$. But $\sigma(se-u) = s - \sigma(u)$, seen as above.
$\Rightarrow \sigma(u) = s - \sigma(se-u) \subset [0, 2s]$. Since $u^* = u$, this implies $u \geq 0$ ∎

"e)" Set $x_0 = xx^* \overset{18.4.a)}{\Rightarrow} x_0^* = x_0$. Choose $E, M, \Delta_M, \mathcal{G}$ as in "a)".
$\Rightarrow \hat{x}_0 : \Delta_M \to \mathbb{C}$ is real-valued, and $\sigma(x_0) = \hat{x}_0(\Delta_M)$.
Thus it suffices to show that $\hat{x}_0(h) \geq 0 \quad \forall h \in \Delta_M$.
By 18.10., $\mathcal{G}(\Delta_M) = \hat{M} = C(\Delta_M)$. Since the map $z \mapsto |z| - z$ is continuous on $\mathbb{C}$ and $\hat{x}_0 \in C(\Delta_M) \Rightarrow f(h) := |\hat{x}_0(h)| - \hat{x}_0(h)$ is contin. on $\Delta_M \Rightarrow \exists y \in M$ s.t. $\hat{y} = |\hat{x}_0| - \hat{x}_0$ on $\Delta_M$.
As $\hat{x}_0(h) \in \mathbb{R} \Rightarrow \hat{y}(h) \in \mathbb{R} \; \forall h \Rightarrow y^* = y$ by 18.10. Set $w = yx \in A \Rightarrow \exists u, v \in A$ s.t. $u^* = u, v^* = v$, and $w = u + iv$ (18.4.b)
$\Rightarrow ww^* = yxx^*y^* = yx_0y = y^2x_0$ as $M$ is commutative.
But $ww^* = (u+iv)(u-iv) = u^2 - iuv + ivu + v^2$
$w^*w = (u-iv)(u+iv) = u^2 - ivu + iuv + v^2$
$= -ww^* + u^2 + v^2 + u^2 + v^2 = 2u^2 + 2v^2 - y^2x_0$.
Since $u^* = u \overset{a)}{\Rightarrow} \sigma(u) \subset \mathbb{R}$. Also $\lambda^2 e - u^2 = (\lambda e - u)(\lambda e + u)$
$= (\lambda e + u)(\lambda e - u)$, and thus $\lambda \notin \mathbb{R} \Rightarrow \lambda, -\lambda \notin \mathbb{R} \supset \sigma(u)$
$\Rightarrow \lambda e + u,\ -((-\lambda)e - u) = \lambda e + u \in G(A) \Rightarrow \lambda^2 e - u^2 \in G(A) \Rightarrow \lambda^2 \notin \sigma(u^2)$.
Thus $\sigma(u^2) \subset [0, \infty)$, $\Rightarrow u^2 \geq 0$, as $(u^2)^* = u^*u^* = u^2$.
$\overset{d)}{\Rightarrow} 2u^2 = u^2 + u^2 \geq 0$. Similarly, $v^* = v \Rightarrow 2v^2 \geq 0 \overset{d)}{\Rightarrow} 2u^2 + 2v^2 \geq 0$.
Finally, since $y, x_0 \in M \Rightarrow -y^2x_0 \in M$ and $\mathcal{G}(-y^2x_0) = -\hat{y}^2\hat{x}_0$.
If we denote $c := \hat{x}_0(h)$ for some $h \in \Delta_M$, then $c \in \mathbb{R}$ and

$$\mathcal{G}(-y^2x_0)(h) = -(|c|-c)^2 c = \begin{cases} 0, & \text{if } c \geq 0 \\ -(-c-c)^2 c = (-c)(-2c)^2 = 4|c|^3, & \text{if } c < 0 \end{cases}$$

$\Rightarrow \mathcal{G}(-y^2x_0)(\Delta_M) \subset [0, \infty) \overset{17.5.e)}{\Rightarrow} \sigma(-y^2x_0) \subset [0, \infty) \Rightarrow w^*w = (2u^2+2v^2) + (-y^2x_0)$
is positive, by "d)". ($(-y^2x_0)^* = -x_0^*y^*y^* = -y^2x_0$.). Suppose
$\lambda \neq 0$ and $\lambda \in \rho(w^*w) \Rightarrow \lambda e - w^*w \in G(A) \overset{\lambda \neq 0}{\Rightarrow} e - \lambda^{-1}w^*w \in G(A)$.
Set $z := e + \lambda^{-1}w(e - \lambda^{-1}w^*w)^{-1}w^* \Rightarrow z(e - \lambda^{-1}ww^*) = z - \lambda^{-1}ww^*$
$- \lambda^{-1}w(e - \lambda^{-1}w^*w)^{-1}w^*(\lambda^{-1}ww^*) = z - \lambda^{-1}w\cancel{w^*}\lambda^{-1}w(e - \lambda^{-1}w^*w)^{-1}(\lambda^{-1}w^*w - e + e)w^*$
$= z - \lambda^{-1}ww^* - \lambda^{-1}w(-w^*) - \lambda^{-1}w(e - \lambda^{-1}w^*w)^{-1}w^* = e$, and
$(e - \lambda^{-1}ww^*)z = z - \lambda^{-1}ww^* - \lambda^{-1}w\underbrace{w^*(\lambda^{-1}w)(e - \lambda^{-1}w^*w)^{-1}}_{= (\lambda^{-1}w^*w - e + e)(\ )^{-1} = -e + (\ )^{-1}}w^*$
$= z - \cancel{\lambda^{-1}ww^*} + \lambda^{-1}ww^* - \lambda^{-1}w(e - \lambda^{-1}w^*w)^{-1}w^* = e$.
Thus $e - \lambda^{-1}ww^* \in G(A) \Rightarrow \lambda e - ww^* \in G(A) \Rightarrow \lambda \notin \sigma(ww^*)$.
Therefore, $\sigma(ww^*) \subset \sigma(w^*w) \cup \{0\} \subset [0, \infty)$, $\Rightarrow \mathcal{G}(y^2x_0)(h) \geq 0 \; \forall h$
$\Rightarrow 0 \leq \mathcal{G}(-y^2x_0)(h) = -\mathcal{G}(y^2x_0)(h) \leq 0 \Rightarrow 0 = \mathcal{G}(+y^2x_0)(h) = -(|c|-c)^2 c$
$\Rightarrow c = 0$ or $c = |c| \Rightarrow c = |c| \Rightarrow \hat{x}_0(h) = |\hat{x}_0(h)| \geq 0 \; \forall h \in \Delta_M$
$\therefore xx^* \geq 0 \Rightarrow x^*x = x^*(x^*)^* \geq 0$. ∎

"f)" Let $y = x^*x$ or $xx^*$, By e) $\Rightarrow \sigma(y) \subset [0,\infty)$
$\Rightarrow -1 \notin \sigma(y) \Rightarrow -e-y = -(e+y) \in G(A) \Rightarrow e+y \in G(A)$. $\square$

**18.14. Theorem**: Suppose $A$ is a $C^*$-algebra and $A_0 \subset A$ is a closed subalgebra for which $e \in A_0$ and $x^* \in A_0$ $\forall x \in A_0$. Then $\sigma_A(x) = \sigma_{A_0}(x)$ $\forall x \in A_0$.

Proof. As in the proof of 18.7., it suffices to show that $G(A_0) \supset G(A) \cap A_0$. (since $x \in A_0 \Rightarrow \lambda e - x \in A_0$)
Suppose $x \in A_0 \cap G(A) \overset{18.4.d)}{\Rightarrow} x^* \in A_0 \cap G(A) \Rightarrow x^*x \in A_0 \cap G(A)$
$\Rightarrow 0 \notin \sigma_A(x^*x) \overset{18.13.e)}{\Rightarrow} \sigma_A(x^*x) \subset (0,\infty) \Rightarrow \rho_A(x^*x)$ is connected
$\overset{16.17.d)}{\Rightarrow} \sigma_A(x^*x) = \sigma_{A_0}(x^*x) \Rightarrow 0 \notin \sigma_{A_0}(x^*x) \Rightarrow x^*x \in G(A_0)$
$\Rightarrow x^{-1} = x^{-1}(x^*)^{-1}x^* = (x^*x)^{-1}x^* \in A_0 \Rightarrow x \in G(A_0)$ $\square$

**18.15. Corollary**: In the statements of theorem 18.11.,
$\sigma(x) := \sigma_{A_0}(x) = \sigma_{B(H)}(x)$.

Proof.: $A_0$ is $C^*$-subalgebra, $e \in A_0 \Rightarrow$ can apply 18.14. $\square$

## 19. Spectral theory on $B(H)$

**19.1. Definition**: Suppose $M$ is a $\sigma$-algebra in a set $X$, and $H$ is a Hilbert space. A mapping $E: M \to B(H)$ is a resolution of the identity if it satisfies

a) $E(\emptyset) = 0$, $E(X) = 1 = \mathrm{id}_H$
b) $E(\omega)$ is a self-adjoint projection $\forall \omega \in M$
c) $E(\omega \cap \omega') = E(\omega)E(\omega')$ $\forall \omega, \omega' \in M$
d) If $\omega, \omega' \in M$ and $\omega \cap \omega' = \emptyset$, then $E(\omega \cup \omega') = E(\omega) + E(\omega')$.
e) $\forall \psi, \varphi \in H$ the map $E_{\varphi,\psi}: M \to \mathbb{C}$ defined by

$$E_{\varphi,\psi}(\omega) := (\varphi, E(\omega)\psi)$$

is a complex measure on $M$.

If $X$ = locally compact Hausdorff space and $M$ = Borel $\sigma$-algebra on $X$, we say that $E$ is a regular resolution of the identity, if every $E_{\varphi,\psi}$ is a regular Borel measure (see 2.15 and 6.19 in [ECA])

<u>19.2. Proposition:</u> Suppose $\mathcal{H}$ is a Hilbert space, $\mathcal{M}$ is a $\sigma$-algebra in $\mathbb{X}$, and $E : \mathcal{M} \to \mathcal{B}(\mathcal{H})$ is a resolution of the identity. Define $E_{\varphi,\psi}$ by e) above for $\varphi, \psi \in \mathcal{H}$. Consider an arbitrary $\psi_0 \in \mathcal{H}$. Then

a) $E_{\psi_0,\psi_0}$ is a bounded positive measure on $\mathcal{M}$ whose total variation is $\|\psi_0\|^2$, and $E_{\psi_0,\psi_0}(\omega) = \|E(\omega)\psi_0\|^2 \ \forall \omega \in \mathcal{M}$.

b) $E(\omega)$ and $E(\omega')$ commute $\forall \omega, \omega' \in \mathcal{M}$

c) If $\omega, \omega' \in \mathcal{M}$ are disjoint, then $R(E(\omega)) \perp R(E(\omega'))$.

d) $E$ is <u>finitely</u> additive.

e) the map $\omega \mapsto E(\omega)\psi_0$ is a countably additive $\mathcal{H}$-valued measure on $\mathcal{M}$.

f) If $\omega_n \in \mathcal{M}$, $n \in \mathbb{N}_+$, all satisfy $E(\omega_n) = 0$, then $E(\bigcup_{n\in\mathbb{N}_+} \omega_n) = 0$.

<u>Proof:</u> "a)" If $\omega \in \mathcal{M} \Rightarrow E(\omega)^* = E(\omega) = E(\omega)^2 \Rightarrow$

$$E_{\psi_0,\psi_0}(\omega) = (\psi_0, E(\omega)\psi_0) = (\psi_0, E(\omega)^* E(\omega)\psi_0)$$
$$= (E(\omega)\psi_0, E(\omega)\psi_0) = \|E(\omega)\psi_0\|^2 \geq 0$$

$\Rightarrow E_{\psi_0,\psi_0}(\mathbb{X}) = \|\psi_0\|^2 < \infty$. Thus $E_{\psi_0,\psi_0}$ is a bounded positive measure with $\|E_{\psi_0,\psi_0}\| = \|\psi_0\|^2$. ∎

"b)" If $\omega, \omega' \in \mathcal{M} \overset{c)}{\Rightarrow} E(\omega)E(\omega') = E(\omega\cap\omega') = E(\omega'\cap\omega) = E(\omega')E(\omega)$.

"c)" If $\omega \cap \omega' = \emptyset \Rightarrow 0 \overset{a)}{=} E(\emptyset) = E(\omega\cap\omega') \overset{c)}{=} E(\omega)E(\omega')$

$\Rightarrow \forall \varphi, \psi \in \mathcal{H}: \ 0 = (\varphi, E(\omega)E(\omega')\psi) = (E(\omega)^*\varphi, E(\omega')\psi)$

$= (E(\omega)\varphi, E(\omega')\psi) \Rightarrow R(E(\omega)) \perp R(E(\omega'))$.

"d) & e)" If $\omega_n \in \mathcal{M}$, $n \in \mathbb{N}_+$, are <u>disjoint</u>, then $\forall N \in \mathbb{N}_+$

$$\omega_{N+1} \cap \left(\bigcup_{n=1}^{N} \omega_n\right) = \bigcup_{n=1}^{N} (\omega_{N+1} \cap \omega_n) = \emptyset \Rightarrow$$

$$E\left(\bigcup_{n=1}^{N+1} \omega_n\right) \overset{d)}{=} E(\omega_{N+1}) + E\left(\bigcup_{n=1}^{N} E_n\right).$$ Thus by induction

$$\Rightarrow E\left(\bigcup_{n=1}^{N} \omega_n\right) = \sum_{n=1}^{N} E(\omega_n) \quad \forall N \in \mathbb{N}_+.$$ Proves "d)".

(typically <u>not</u> countably additive, since by Ex. 3.4. here either $E(\omega_n) = 0$ or $\|E(\omega_n)\| = 1 \Rightarrow$ the sum is not norm-Cauchy unless $E(\omega_n) = 0 \ \forall n \geq n_0$.)

To prove "e)" it suffices to show that now

$$E\left(\bigcup_{n\in\mathbb{N}_+} \omega_n\right)\psi_0 = \sum_{n=1}^{\infty} E(\omega_n)\psi_0$$ as an $\mathcal{H}$-convergent sum.

For this, denote $\varphi_n := E(\omega_n)\psi_0 \in R(E(\omega_n))$. By "c)" $\Rightarrow \varphi_n \perp \varphi_m \quad \forall n \neq m. \Rightarrow$ if $I \subset \mathbb{N}_+$ and $|I| < \infty$ then

$$\Big\|\sum_{n\in I} \varphi_n\Big\|^2 = \sum_{n',n\in I} (\varphi_{n'}, \varphi_n) = \sum_{n\in I} \|\varphi_n\|^2$$

$$= \sum_{n\in I} (E(\omega_n)\psi_0, E(\omega_n)\psi_0) = \sum_{n\in I} (\psi_0, E(\omega_n)\psi_0)$$

$$= \sum_{n\in I} E_{\psi_0,\psi_0}(\omega_n) = E_{\psi_0,\psi_0}\Big(\bigcup_{n\in I}\omega_n\Big) \overset{\text{"a)"}}{\leq} \|\psi_0\|^2$$

$$\Rightarrow \sum_{n\in\mathbb{N}_+} \|\varphi_n\|^2 \leq \|\psi_0\|^2 \Rightarrow \forall \varepsilon > 0 \ \exists n_0 \text{ s.t. } \forall m \geq 0$$

$$\Big\|\sum_{n=n_0}^{n_0+m} \varphi_n\Big\|^2 = \sum_{n=n_0}^{n_0+m} \|\varphi_n\|^2 \leq \varepsilon^2 \Rightarrow \Big(\sum_{n=1}^{N}\varphi_n\Big)_{N\in\mathbb{N}_+} \text{ is}$$

Cauchy in $\mathcal{H}$ $\Rightarrow \exists \tilde{\psi} = \lim_{N\to\infty} \sum_{n=1}^{N} \varphi_n. \Rightarrow \forall \psi \in \mathcal{H}$:

$$(\psi, \tilde{\psi}) = \lim_{N\to\infty} \sum_{n=1}^{N} (\psi, E(\omega_n)\psi_0) = \sum_{n=1}^{\infty} E_{\psi,\psi_0}(\omega_n)$$

$$= E_{\psi,\psi_0}\Big(\bigcup_{n=1}^{\infty}\omega_n\Big) = (\psi, E(\cup\omega_n)\psi_0)$$

$$\Rightarrow \tilde{\psi} = \sum_{n=1}^{\infty} E(\omega_n)\psi_0 = E\Big(\bigcup_{n=1}^{\infty}\omega_n\Big)\psi_0 \quad \blacksquare$$

"f)" Let $\omega := \bigcup_{n=1}^{\infty}\omega_n \in \mathcal{M}$. Then $\forall \psi \in \mathcal{H}$, $E_{\psi,\psi}$ is a measure

$$\overset{\text{"a)"}}{\Rightarrow} \|E(\omega)\psi\|^2 = E_{\psi,\psi}(\omega) = \int E_{\psi,\psi}(d\lambda)\, \mathbb{1}(\lambda\in\omega)$$

$$\overset{\text{DCT}}{=} \lim_{N\to\infty} \int E_{\psi,\psi}(d\lambda)\, \underbrace{\mathbb{1}\Big(\lambda\in\bigcup_{n=1}^{N}\omega_n\Big)}_{\leq \sum_{n=1}^{N}\mathbb{1}(\lambda\in\omega_n)} \leq \sum_{n=1}^{\infty} \int E_{\psi,\psi}(d\lambda)\, \mathbb{1}(\lambda\in\omega_n)$$

$\Rightarrow \|E(\omega)\psi\|^2 \leq \sum_{n=1}^{\infty} E_{\psi,\psi}(\omega_n)$, where $E_{\psi,\psi}(\omega_n)$ $= (\psi, E(\omega_n)\psi) = 0$ as $E(\omega_n) = 0$.

$\Rightarrow 0 \leq \|E(\omega)\psi\|^2 \leq 0 \Rightarrow E(\omega)\psi = 0$. $\therefore E(\omega) = 0 \ \square$

* Because each $E(\omega)$ is a projection and by e) above $E(\omega)\psi$ is an $\mathcal{H}$-valued measure, these $E$ are also called <u>projection valued measures</u> (PVM).

## 19.3. Theorem (Spectral theorem I)

Suppose $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is a subalgebra which is also closed and normal in $\mathcal{B}(\mathcal{H})$, and which contains the identity operator $\mathbb{1} = id_{\mathcal{H}}$. Then $\mathcal{A}$ is a commutative $C^*$-algebra; let $\Delta$ denote corresponding maximal ideal space of $\mathcal{A}$. In addition,

a) $\exists!$ regular resolution of the identity $E$ on the Borel $\sigma$-algebra of $\Delta$ for which

$$(*) \qquad (\varphi, T\psi) = \int_\Delta E_{\varphi,\psi}(d\lambda)\, \hat{T}(\lambda) \qquad \forall T \in \mathcal{A}, \varphi,\psi \in \mathcal{H}$$

where $\hat{T} = \mathcal{G}(T) =$ Gelfand transform of $T$.

b) If $\omega \subset \Delta$ is open and nonempty, then $E(\omega) \neq 0$.

c) Consider some $S \in \mathcal{B}(\mathcal{H})$. Then $ST = TS\ \forall T \in \mathcal{A}$ iff $SE(\omega) = E(\omega)S\ \forall \omega \in M$.

* Typically, $(*)$ is abbreviated by $T = \int_\Delta dE\, \hat{T}$.

**Proof.** Since $\mathcal{A}$ is a closed normal algebra and $\mathbb{1} \in \mathcal{A}$, it is equal to the Banach subalgebra generated by it.

18.5. $\Rightarrow$ it is a commutative $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$.

By 18.10. the Gelfand transform $\mathcal{G}: \mathcal{A} \to C(\Delta)$ is an isometric $*$-isomorphism with $\mathcal{G}(\mathcal{A}) = C(\Delta)$.

By 17.8. $\Delta$ is a compact Hausdorff space $\Rightarrow$ every $f \in C(\Delta)$ has a compact support. If $\varphi, \psi \in \mathcal{H}$ then $f \mapsto (\varphi, \mathcal{G}^{-1}(f)\psi)$ is a linear map $C(\Delta) \to \mathbb{C}$ and $|(\varphi, \mathcal{G}^{-1}(f)\psi)| \leq \|\varphi\|\,\|\psi\|\,\|\mathcal{G}^{-1}(f)\| \overset{\mathcal{G}\text{ isom.}}{=} \|\varphi\|\,\|\psi\|\,\|f\|_\infty$

$\Rightarrow$ the map has norm $\|\varphi\|\,\|\psi\| < \infty$ $\Rightarrow$ it is bounded.

By Riesz representation theorem ([RCA, 6.19])

$\Rightarrow \exists!$ regular Borel measure $E_{\varphi,\psi}$ on $\Delta$ for which $(\varphi, \mathcal{G}^{-1}(f)\psi) = \int_\Delta E_{\varphi,\psi}(d\lambda)\, f(\lambda)\quad \forall f \in C(\Delta)$.

$\Rightarrow \forall T \in \mathcal{A}: (\varphi, T\psi) = \int_\Delta E_{\varphi,\psi}(d\lambda)\, \hat{T}(\lambda) \qquad (f = \mathcal{G}(T) = \hat{T})$

$\Rightarrow$ $(*)$ holds for $E_{\varphi,\psi}$. If $\tilde{E}$ is a regular resol. of identity for which $(*)$ holds $\Rightarrow$ $\tilde{E}_{\varphi,\psi}$ is a regular Borel measure, and thus by Riesz $\Rightarrow \tilde{E}_{\varphi,\psi} = E_{\varphi,\psi}\ \forall \varphi,\psi$.

$\Rightarrow \forall \omega \in M, \varphi,\psi \in \mathcal{H}: (\varphi, \tilde{E}(\omega)\psi) = \tilde{E}_{\varphi,\psi}(\omega) = E_{\varphi,\psi}(\omega)$

Thus if $\exists E(\omega)$ s.t. $E_{\varphi,\psi} = (\varphi, E(\omega)\psi)$
$\Rightarrow \tilde{E}(\omega)\psi = E(\omega)\psi \;\; \forall \psi, \omega \Rightarrow \tilde{E}(\omega) = E(\omega) \;\; \forall \omega \in M.$
This proves uniqueness in "a)".

The Riesz represent. theorem also implies the the total variation norm of $E_{\varphi,\psi}$ is equal to the norm of the linear map $\Rightarrow \|E_{\varphi,\psi}\| := |E_{\varphi,\psi}|(X) \leq \|\varphi\|\,\|\psi\|$.

Suppose $T \in A$ is such that $\hat{T}$ is real-valued $\overset{18.10}{\Rightarrow}$ $T$ is self-adjoint $\Rightarrow (\varphi, T\psi) = (T^*\varphi, \psi) = (\psi, T^*\varphi)^* \;\; \forall \psi, \varphi \in \mathcal{H}$.

$\Rightarrow \int_\Delta E_{\varphi,\psi}(d\lambda)\, \hat{T}(\lambda) = \left( \int_\Delta E_{\psi,\varphi}(d\lambda)\, \underbrace{G(T^*)(\lambda)}_{= \hat{T}(\lambda)^*} \right)^*$

$= \int_\Delta |E_{\psi,\varphi}|(d\lambda)\, h_{\psi,\varphi}(\lambda)^*\, \hat{T}(\lambda) \qquad \forall \hat{T}$, real-valued

$\hat{A} = C(\Delta)$

$\overset{\hat{A}=C(\Delta)}{\Rightarrow} \int_\Delta E_{\varphi,\psi}(d\lambda)\, f(\lambda) = \int_\Delta E^*_{\psi,\varphi}(d\lambda)\, f(\lambda) \quad \forall f \in C(\Delta)$

$\Rightarrow \forall \varphi,\psi : E^*_{\psi,\varphi} = E_{\varphi,\psi}$ by uniq. in Riesz

$\left( E^*_{\varphi,\psi}(\omega) := E_{\varphi,\psi}(\omega)^* = \left( \int_\omega E_{\varphi,\psi}(d\lambda) \right)^* = \int_\omega |E_{\varphi,\psi}|(d\lambda)\, h_{\varphi,\psi}(\lambda)^* \right)$

Suppose $\omega \in M$ = Borel $\sigma$-algebra of $\Delta$. If $\varphi, \psi, \psi' \in \mathcal{H}$ and $\alpha, \alpha' \in \mathbb{C} \Rightarrow (\varphi, T(\alpha\psi + \alpha'\psi')) = \alpha(\varphi, T\psi) + \alpha'(\varphi, T\psi')$

$\Rightarrow \forall f \in C(\Delta): \int_\Delta E_{\varphi,\alpha\psi+\alpha'\psi'}(d\lambda)\, f(\lambda) = \alpha \int_\Delta E_{\varphi,\psi}(d\lambda)\, f(\lambda)$
$+ \alpha' \int_\Delta E_{\varphi,\psi'}(d\lambda)\, f(\lambda) \underset{\text{Lusin's theorem}}{\Rightarrow}$ true for all Borel meas. $f$ with $\|f\|_\infty < \infty$.

Thus if $f$ is Borel meas. and $\|f\|_\infty < \infty$, the map $(\varphi, \psi) \mapsto \int_\Delta E_{\varphi,\psi}(d\lambda)\, f(\lambda)$ is linear. $\Rightarrow$ conj. lin. in $\varphi$, as $\int_\Delta E_{\varphi,\psi}(d\lambda)\, f(\lambda) = \left( \int_\Delta E_{\psi,\varphi}(d\lambda)\, f(\lambda)^* \right)^*$
$\Rightarrow$ it is sesquilin. Also $\left| \int_\Delta E_{\varphi,\psi}(d\lambda) f(\lambda) \right| \leq \|f\|_\infty \int_\Delta |E_{\varphi,\psi}|(d\lambda)$
$= \|f\|_\infty \|E_{\varphi,\psi}\| \leq \|f\|_\infty \|\varphi\|\,\|\psi\| \quad \forall \varphi, \psi \in \mathcal{H}$.

Thus by 3.1.b) $\Rightarrow \exists!\, \Phi(f) \in \mathcal{B}(\mathcal{H})$ s.t. $\|\Phi(f)\| \leq \|f\|_\infty$ and $\int_\Delta E_{\varphi,\psi}(d\lambda)\, f(\lambda) = (\varphi, \Phi(f)\psi) \quad \forall \varphi, \psi \in \mathcal{H}$.

For $f \in C(\Delta)$, it follows that $\Phi(f) = G^{-1}(f)$. In addition, since $(\Phi(f^*)\varphi, \psi) = (\psi, \Phi(f^*)\varphi)^* = \left( \int E_{\psi,\varphi}(d\lambda)\, f(\lambda)^* \right)^*$
$= \int E^*_{\psi,\varphi}(d\lambda)\, f(\lambda) = \int E_{\varphi,\psi}(d\lambda)\, f(\lambda) = (\varphi, \Phi(f)\psi)$
$\Rightarrow \Phi(f)^* = \Phi(f^*) \;\; \forall f$. In particular, if $f$ is real-valued
$\Rightarrow \Phi(f)^* = \Phi(f)$, i.e. $\Phi(f)$ is self-adjoint.

If $S, T \in A$, then $G(ST) = G(S)G(T) \Rightarrow \forall \varphi, \psi \in \mathcal{H}$:
$\int_\Delta E_{\varphi,\psi}(d\lambda)\, \hat{S}(\lambda)\hat{T}(\lambda) = \int E_{\varphi,\psi}(d\lambda)\, (ST)\hat{\ }(\lambda) = (\varphi, ST\psi)$
$= \int E_{\varphi,T\psi}(d\lambda)\, \hat{S}(\lambda)$. Since $\hat{S} \in C(\Delta)$ can be arbitrary
$\Rightarrow E_{\varphi,T\psi}(d\lambda) = E_{\varphi,\psi}(d\lambda)\, \hat{T}(\lambda) \quad \forall T \in A$.

Again for arbitrary $f$ as above, $\Rightarrow$

$$\int_\Delta E_{\varphi,\psi}(d\lambda)\, f(\lambda)\hat{T}(\lambda) = \int_\Delta E_{\varphi, T\psi}(d\lambda)\, f(\lambda)$$

$$= (\varphi, \Phi(f)T\psi) = (\Phi(f)^*\varphi, T\psi) = \int_\Delta E_{\Phi(f^*)\varphi,\psi}(d\lambda)\,\hat{T}(\lambda)$$

$$\Rightarrow E_{\varphi,\psi}(d\lambda)\, f(\lambda) = E_{\Phi(f^*)\varphi,\psi}(d\lambda)$$

$\Rightarrow \forall f, g$ Borel meas. & $\|f\|_\infty, \|g\|_\infty < \infty$

we have $\int_\Delta E_{\varphi,\psi}(d\lambda)\, \underbrace{f(\lambda)g(\lambda)}_{=(fg)(\lambda)} = \int_\Delta E_{\Phi(f^*)\varphi,\psi}(d\lambda)\, g(\lambda)$

$\Rightarrow (\varphi, \Phi(fg)\psi) = (\Phi(f)^*\varphi, \Phi(g)\psi) = (\varphi, \Phi(f)\Phi(g)\psi)$ $\forall \varphi,\psi$

$\therefore \Phi(fg) = \Phi(f)\Phi(g)$.

Define $E(\omega) = \Phi(\mathbb{1}(\lambda\in\omega))$, where the characteristic function $\mathbb{1}(\lambda\in\omega) \in [0,1]$ and it is Borel measurable.

If $\omega', \omega \in \mathcal{M}$, then $\mathbb{1}(\lambda \in \omega\cap\omega') = \begin{cases} 0, & \text{if } \lambda\notin\omega \text{ or } \lambda\notin\omega' \\ 1, & \text{if } \lambda\in\omega \text{ and } \lambda\in\omega' \end{cases}$

$= \mathbb{1}(\lambda\in\omega)\mathbb{1}(\lambda\in\omega')$

$\Rightarrow E(\omega'\cap\omega) = E(\mathbb{1}(\lambda\in\omega)\mathbb{1}(\lambda\in\omega')) = E(\mathbb{1}(\lambda\in\omega))E(\mathbb{1}(\lambda\in\omega'))$

$= E(\omega)E(\omega')$. $\Rightarrow E(\omega) = E(\omega)^2$ $(\omega' = \omega)$

As $\mathbb{1}(\lambda\in\omega)$ is real, also $E(\omega)^* = E(\omega)$ $\therefore E(\omega)$ is a self-adjoint projection $\forall \omega\in\mathcal{M}$. $E(\emptyset) = 0$, since $0 = \int_\Delta E_{\varphi,\psi}(d\lambda)\,\mathbb{1}(\lambda\in\emptyset)$ $\forall\varphi,\psi$, and $E(\Delta) = 1$ since $1\in\mathcal{A}$ and $\mathcal{G}(1)(h) = h(1) = 1 = \mathbb{1}(\lambda\in\Delta)$. If $\omega\cap\omega' = \emptyset$

$\Rightarrow \mathbb{1}(\lambda\in\omega\cup\omega') = \mathbb{1}(\lambda\in\omega) + \mathbb{1}(\lambda\in\omega') - \underbrace{\mathbb{1}(\lambda\in\omega\cap\omega')}_{=0}$

$\Rightarrow (\varphi, E(\omega\cup\omega')\psi) = \int_\Delta E_{\varphi,\psi}(d\lambda)\,\mathbb{1}(\lambda\in\omega\cup\omega')$

$= (\varphi, E(\omega)\psi) + (\varphi, E(\omega')\psi)$ $\forall\varphi,\psi$

$\Rightarrow E(\omega\cup\omega') = E(\omega) + E(\omega')$. Finally, $E_{\varphi,\psi}(\omega)$

$= \int_\Delta E_{\varphi,\psi}(d\lambda)\,\mathbb{1}(\lambda\in\omega) = (\varphi, E(\omega)\psi)$ $\forall\varphi,\psi,\omega$.

$\therefore E: \mathcal{M}\to\mathcal{B}(\mathcal{H})$ is a regular resolution of the identity, and $(*)$ is satisfied. Thus "a)" holds. ■

"b)" Suppose then that $\omega\subset\Delta$ is open ($\Rightarrow \omega\in\mathcal{M}$) and $E(\omega) = 0$. If $\lambda_0\in\omega$, by Urysohn's lemma ([RCA, 2.12]) $\exists f\in C(\Delta)$ s.t.

$0\le f(\lambda)\le 1$ $\forall\lambda\in\Delta$, $f(\lambda_0) = 1$, and $f(\lambda) = 0$ $\forall\lambda\notin\omega$.

$\Rightarrow \exists T = \mathcal{G}^{-1}(f)\in\mathcal{A}$, $T\ne 0 = \mathcal{G}(0)$. However, $\forall\psi\in\mathcal{H}$

$\|T\psi\|^2 = (\psi, T^*T\psi) = (\psi, \Phi(f^*)\Phi(f)\psi) = (\psi, \Phi(f^2)\psi)$

$= \int \underset{\text{pos. measure}}{E_{\psi,\psi}}(d\lambda)\, f(\lambda)^2 \le \int E_{\psi,\psi}(d\lambda)\,\mathbb{1}(\lambda\in\omega) = (\psi, E(\omega)\psi) = 0$

$\Rightarrow T = 0$ ↯. Thus $\omega = \emptyset$. ■

"c)" Suppose $S \in \mathcal{B}(\mathcal{H})$, $\varphi, \psi \in \mathcal{H}$, and set $\varphi_0 = S^*\varphi \in \mathcal{H}$.
$\Rightarrow \forall T \in \mathcal{A}$ and $\omega \in \mathcal{M}$:

$$(\varphi, ST\psi) = (S^*\varphi, T\psi) = (\varphi_0, T\psi) = \int_\Delta dE_{\varphi_0,\psi}\, \hat{T}$$

$$(\varphi, TS\psi) = \int_\Delta dE_{\varphi, S\psi}\, \hat{T} \quad \text{and}$$

$$(\varphi, SE(\omega)\psi) = (\varphi_0, E(\omega)\psi) = E_{\varphi_0,\psi}(\omega)$$

$$(\varphi, E(\omega)S\psi) = E_{\varphi,S\psi}(\omega).$$

Thus if $ST = TS \Rightarrow E_{\varphi_0,\psi} = E_{\varphi,S\psi} \Rightarrow (\varphi, SE(\omega)\psi)$
$= (\varphi, E(\omega)S\psi) \Rightarrow SE(\omega) = E(\omega)S$.
If $SE(\omega) = E(\omega)S \Rightarrow E_{\varphi_0,\psi} = E_{\varphi,S\psi}$ on $\mathcal{M}$
$\Rightarrow (\varphi, ST\psi) = (\varphi, TS\psi) \Rightarrow ST = TS$ □

<u>19.4. Corollary</u> (Spectral decomposition of normal operators)

Suppose $T \in \mathcal{B}(\mathcal{H})$ is normal. Then $\exists!$ regular resolution of the identity $E$ on the Borel $\sigma$-algebra $\mathcal{M}$ of $\sigma(T)$ which satisfies

$$T = \int_{\sigma(T)} E(d\lambda)\, \lambda .$$

In addition, if $S$ commutes with $T$, then $S$ commutes with every $E(\omega)$, $\omega \in \mathcal{M}$.

* This $E$ is called the <u>spectral decomposition</u> of $T$.

* One can trivially extend $E$ to the Borel $\sigma$-algebra of the whole $\mathbb{C}$ by defining $E(\omega) = E(\omega \cap \sigma(T))$.

<u>19.5. Corollary</u> (Simultaneous spectral decomposition)

Suppose $T_i \in \mathcal{B}(\mathcal{H})$, $i = 1, \dots, N$, $N \in \mathbb{N}_+$, are <u>normal</u> operators which <u>commute</u>. Then $\exists K \subset \mathbb{C}^N$, which is a nonempty compact subset and $\lambda \in K \Rightarrow \lambda_i \in \sigma(T_i)\ \forall i = 1, \dots, N$, such that $\exists!$ regular resolution of the identity on the Borel $\sigma$-algebra $\mathcal{M}$ of $K$ with $T_i = \int_K E(d\lambda)\, \lambda_i\ \forall i = 1, \dots, N$.
If $S$ commutes with every $T_i$, $i = 1, \dots, N$, then $S$ commutes with every $E(\omega)$, $\omega \in \mathcal{M}$.

<u>Proof of 19.4. & 19.5.</u> : Let us first prove 19.5.
Denote $E_0 := \{T_i \mid i=1,\dots,N\}$.
By assumption, $E_0$ commutes and its every element is normal. By 18.11. the set $E = E_0 \cup E_0^*$ is normal, and the set $A_0 := \overline{\text{span}(F)}$, with $F = \{\text{monomials of } E\}$, is a closed normal subalgebra of $\mathcal{B}(\mathcal{H})$, and $1 \in F \Rightarrow 1 \in A_0$.
By 19.3. $\exists$ regular resolution of the identity $E_0$ on the Borel $\sigma$-algebra $\mathcal{M}_0$ of the maximal ideal space $\Delta_0$ of $A_0$, such that $T = \int_{\Delta_0} dE_0 \hat{T}$ $\forall T \in A_0$.
By 18.11. the map $\Phi : \Delta_0 \to \mathbb{C}^N$ def. by $\Phi(h)_i := h(T_i)$ is a homeomorphism onto $K := \Phi(\Delta_0) \subset \prod_{i=1}^{N} \sigma(T_i)$, where $K$ is compact. Thus it satisfies the properties stated in the Theorem. ($\Delta_0 \neq \emptyset \Rightarrow K \neq \emptyset$) Let $\mathcal{M}$ denote the Borel $\sigma$-algebra of $K$. Since $\Phi$, and $\Phi^{-1}$ are continuous $\Rightarrow$ Borel measurable $\Rightarrow$ if $\omega \in \mathcal{M}_0$, then $\Phi(\omega) = (\Phi^{-1})^{\leftarrow}(\omega) \in \mathcal{M}$ [Rudin, RCA, 1.12.b] and, similarly, if $\omega' \in \mathcal{M}$ then $\Phi^{-1}(\omega') = \Phi^{\leftarrow}(\omega') \in \mathcal{M}_0$. Thus $\omega \mapsto \Phi(\omega)$ is a bijection $\mathcal{M}_0 \to \mathcal{M}$. We define $E(\omega) := E_0(\Phi^{-1}(\omega))$. Then $E : \mathcal{M} \to \mathcal{B}(\mathcal{H})$ is a regular resolution of the identity. [Proof: (counting items in 19.1.) $\Phi^{-1}(\emptyset) = \emptyset$ and $\Phi^{-1}(K) = \Delta_0 \Rightarrow$ a) holds. $E_0(\omega')$ is a self-adjoint projection $\forall \omega' \Rightarrow$ b). If $\omega, \omega' \in \mathcal{M}$ $\Rightarrow$ $\Phi^{-1}(\omega) \cap \Phi^{-1}(\omega') = \Phi^{-1}(\omega \cap \omega') \Rightarrow$ c). Also $\Phi^{-1}(\omega \cup \omega') = \Phi^{-1}(\omega) \cup \Phi^{-1}(\omega') \Rightarrow$ d). Fix $\varphi, \psi \in \mathcal{H}$, and denote $\mu_0 := (E_0)_{\varphi,\psi}$ and $\mu := E_{\varphi,\psi}$. $\Rightarrow \mu_0$ is a regular Borel measure. If $\omega \in \mathcal{M}$, then $\mu(\omega) = (\varphi, E(\omega)\psi) = (\varphi, E_0(\Phi^{-1}(\omega))\psi) = \mu_0(\Phi^{-1}(\omega))$.
Thus in $\omega_n \in \mathcal{M}$, $n \in \mathbb{N}_+$, are disjoint, then $\mu(\bigcup_{n=1}^{\infty} \omega_n) = \mu_0(\bigcup_{n=1}^{\infty} \Phi^{-1}(\omega_n)) = \sum_{n=1}^{\infty} \mu_0(\Phi^{-1}(\omega_n)) = \sum_{n=1}^{\infty} \mu(\omega_n)$, and since $\mu : \mathcal{M} \to \mathbb{C}$ it is thus a Borel measure on $K$.
Since $\Phi$ is homeom., $V$ is open in $K \Leftrightarrow \Phi^{-1}(V)$ is open in $\Delta_0$ and $S$ is compact in $K \Leftrightarrow \Phi^{-1}(S)$ is compact in $\Delta_0$. Since $\mu_0$ is a complex measure $\Rightarrow$ bounded $\Rightarrow |\mu_0|(\Delta_0) < \infty$. By def. $|\mu|(\omega) := \sup_{(\omega_n) \text{ partitions } \omega} \sum_n |\mu(\omega_n)|$, and [$\omega_n \in \mathcal{M}$, $n \in \mathbb{N}_+$, are disjoint and $\omega = \bigcup_{n=1}^{\infty} \omega_n$] $\Leftrightarrow$ [$\Phi^{-1}(\omega_n) \in \mathcal{M}_0$, $n \in \mathbb{N}_+$, are disjoint and $\Phi^{-1}(\omega) = \bigcup_{n=1}^{\infty} \Phi^{-1}(\omega_n)$]
$\Rightarrow |\mu|(\omega) = \sup_{(\omega_n') \text{ partitions } \Phi^{-1}(\omega)} \sum_n |\mu_0(\omega_n')| = |\mu_0|(\Phi^{-1}(\omega))$ $\forall \omega \in \mathcal{M}$.

$\Rightarrow |\mu|(K) = |\mu_0|(\Delta_0) < \infty$ and it suffices to check that $\forall \omega \in \mathcal{M}$
$|\mu|(\omega) = \inf\{|\mu|(V) \mid \omega \subset V,\ V \text{ open}\}$ and
$|\mu|(\omega) = \sup\{|\mu|(S) \mid S \subset \omega,\ S \text{ compact}\}$. These however

follow now immediately from $|\mu|(\omega) = |\mu_0|(\Phi^{-1}(\omega))$ since
$\Phi$ homeom. and $\mu_0$ is regular $\Rightarrow |\mu_0|(\omega') = \inf\{\dots\} = \sup\{\dots\}\ \forall \omega' \in M_0$.
Thus $\mu$ is a regular complex Borel measure $\Rightarrow$ e) $\blacksquare$ ]

By 18.11., $T_i = G^{-1}(f_i \circ \Phi)\ \forall i = 1,\dots,N$, where $f_i(\lambda) = \lambda_i$
$\forall \lambda \in K$. $\Rightarrow \forall \varphi, \psi \in \mathcal{H}: (\varphi, T_i \psi) \overset{12.3.}{=} \int_\Delta (E_0)_{\varphi,\psi}(dh)\, \hat{T}_i(h)$
$= \int_\Delta (E_0)_{\varphi,\psi}(dh)\, f_i(\Phi(h)) \overset{*}{=} \int_K E_{\varphi,\psi}(d\lambda)\, f_i(\lambda)$, $(\lambda = \Phi(h))$

[ Since for any Borel measurable $f$, $\exists$ "simple functions" $s_n$, $n \in \mathbb{N}_+$, i.e., $\exists \alpha_{j,n} \in \mathbb{C}$, $A_{j,n} \in M$ $\forall n \in \mathbb{N}_+, j \in \{1,\dots,n\}$, s.t. $s_n(\lambda) = \sum_{j=1}^{n} \alpha_{j,n} \mathbb{1}(\lambda \in A_{j,n})$ and $\|s_n\|_\infty \le \|f\|_\infty$, $s_n(\lambda) \to f(\lambda)\ \forall \lambda \in K$. (For instance, apply [RCA, 1.17] to $(\mathrm{Re} f)_+$, $(\mathrm{Re} f)_-$, $(\mathrm{Im} f)_+$, and $(\mathrm{Im} f)_-$.) Thus by dominated convergence, $\int_\Delta (E_0)_{\varphi,\psi}(dh)\, f_i(\Phi(h))$

$$= \lim_{n\to\infty}\Big(\sum_{j=1}^{n} \alpha_{j,n} \int_\Delta (E_0)_{\varphi,\psi}(dh)\, \overbrace{\mathbb{1}(\Phi(h) \in A_{j,n})}^{= \mathbb{1}(h \in \overbrace{\Phi^{-1}(A_{j,n})}^{\in M_0})}\Big)$$

$$= \lim_{n\to\infty}\Big(\sum_{j=1}^{n} \alpha_{j,n} (E_0)_{\varphi,\psi}(\Phi^{-1}(A_{j,n}))\Big)$$

$$= \lim_{n\to\infty}\Big(\sum_{j=1}^{n} \alpha_{j,n} E_{\varphi,\psi}(A_{j,n})\Big) = \lim_{n\to\infty} \int_K E_{\varphi,\psi}(d\lambda) \sum_{j=1}^{n} \alpha_{j,n} \mathbb{1}(\lambda \in A_{j,n})$$

$$\overset{DCT}{=} \int_K E_{\varphi,\psi}(d\lambda)\, f(\lambda). \blacksquare ]$$

$\Rightarrow (\varphi, T_i \psi) = \int_K E_{\varphi,\psi}(d\lambda)\, \lambda_i\ \forall i, \psi, \varphi$.

$\Rightarrow T_i = \int_K E(d\lambda)\, \lambda_i\ \forall i = 1,\dots,N$. This proves existence of $K, E$.

For uniqueness, consider first $\underline{P} \in \mathrm{span}(F)$, and assume $\tilde{E}$ is another regular resolution of the identity on $M$ with $\int_K \tilde{E}(d\lambda)\lambda_i = T_i$. $\underline{P}$ is a finite polynomial of $e$, $T_i$, and $T_i^*$ and by the following theorem (19.9.) symbolic calculus implies that $T_i^* = \int_K \tilde{E}(d\lambda)\, \lambda_i^* \Rightarrow \underline{P} = \int_K \tilde{E}(d\lambda)\, p(\lambda, \lambda^*)$ where $p$ is a polynomial such that $P = p(T, T^*)$, $T = (T_i)_{i=1}^N$.

Now if $f \in C(K) \overset{18.11.}{\Rightarrow} M := \Psi(f) = G^{-1}(f \circ \Phi) \in A_0 \Rightarrow \exists p_n$ $n \in \mathbb{N}_+$, each a polynomial s.t. $p_n(T, T^*) \xrightarrow{n\to\infty} M$ in $A_0 \overset{18.11.}{\Rightarrow} p_n(\lambda, \lambda^*) = \Psi^{-1}(p_n(T,T^*))$ $\to \Psi^{-1}(M) = f$ in $C(K)$, i.e., in $\|\cdot\|_\infty$-norm. Thus $\forall \varphi, \psi \in \mathcal{H}$
$\int_K \tilde{E}_{\varphi,\psi}(d\lambda)\, f(\lambda) \overset{DCT}{=} \lim_{n\to\infty} \int_K \tilde{E}_{\varphi,\psi}(d\lambda)\, p_n(\lambda, \lambda^*) \overset{SC}{=} \lim_{n\to\infty} (\varphi, p(T, T^*)\psi)$

$\overset{SC}{=} \lim_{n\to\infty} \int_K E_{\varphi,\psi}(d\lambda)\, p_n(\lambda, \lambda^*) = \int_K E_{\varphi,\psi}(d\lambda)\, f(\lambda)$. $\forall f \in C(K)$.
(we applied symbolic calculus twice: first to $\tilde{E}$ then to $E$.)

Thus by Riesz $\Rightarrow \tilde{E}_{\omega,\nu} = E_{\omega,\nu} \Rightarrow \tilde{E} = E$ as on p. 153.
Thus $E$ is unique.

Finally, if $ST_i = T_iS \overset{18.10.''}{\Rightarrow} ST_i^* = T_i^*S$ since $T_i$ is normal.
Thus if $S$ commutes with $E_0 \Rightarrow S$ commutes with $E$
$\Rightarrow S$ commutes with span(F). If $A \in \mathcal{A}_0$, then $\exists A_n \to A$ in $\mathcal{A}_0$, with
$A_n \in$ span(F) $\Rightarrow A_n \to A$ in $\mathcal{B}(\mathcal{H}) \overset{16.2}{\Rightarrow} SA = \lim_{n\to\infty} SA_n = \lim_{n\to\infty} A_nS = AS$

Thus $S$ commutes with $\mathcal{A}_0$
$\overset{19.3.c)}{\Rightarrow} SE_0(\omega') = E_0(\omega')S \quad \forall \omega' \in \mathcal{M}_0$. Since $E_0(\omega') = E(\Phi(\omega'))$
and $\omega' \mapsto \Phi(\omega')$ is a bijection $\mathcal{M}_0 \to \mathcal{M}$, this shows
that $SE(\omega) = E(\omega)S \quad \forall \omega \in \mathcal{M}$. This proves 19.5.

19.4. follows then from 18.11. since $N=1 \Rightarrow K = \sigma(T_1)$ $\square$

## 19.6. Definition: (leading to $L^\infty(E)$)

Suppose $\mathcal{M}$ is a $\sigma$-algebra in a set $\mathfrak{X}$, $\mathcal{H}$ is a Hilbert space, and $E: \mathcal{M} \to \mathcal{B}(\mathcal{H})$ is a resolution of the identity.

Suppose $f: \mathfrak{X} \to \mathbb{C}$ is measurable (i.e. $f^{\leftarrow} U \in \mathcal{M}$ for any $U$ which is open in $\mathbb{C}$).

The $E$-essential range $R_E(f)$ is defined as follows:
$$R_E(f) := \bigcap \{ B(z,\varepsilon)^c \mid \operatorname{Re} z, \operatorname{Im} z \in \mathbb{Q},\ \varepsilon = \tfrac{1}{n} \text{ for some } n \in \mathbb{N}_+, \ \underline{\text{and}}\ E(f^{\leftarrow} B(z,\varepsilon)) = 0 \}$$
Note that such $B(z,\varepsilon)$ form a countable base for the topology of $\mathbb{C}$.

The $E$-essential supremum of $f$ denoted by $\|f\|_{E,\infty}$, is defined by $\|f\|_{E,\infty} := \sup_{z \in R_E(f)} |z|$. If $f: \mathfrak{X} \to \mathbb{C}$ is measurable and $\|f\|_{E,\infty} < \infty$, it is called $E$-essentially bounded.

## 19.7. Proposition: Then

b) $L^\infty_{pre}(E) = \{ f: \mathfrak{X} \to \mathbb{C} \mid f \text{ is measurable and } f(\mathfrak{X}) \text{ is bounded} \}$ is a commutative Banach algebra under $\|f\|_\infty := \sup_{\lambda \in \mathfrak{X}} |f(\lambda)|$.

c) $N := \{ f \in L^\infty_{pre}(E) \mid \|f\|_{E,\infty} = 0 \}$ is a proper closed ideal in $L^\infty_{pre}(E)$.

a) Suppose $f: \mathfrak{X} \to \mathbb{C}$ is measurable. Then $R_E(f) \neq \emptyset$, $R_E(f) \subset \overline{f(\mathfrak{X})}$, and it is the smallest closed subset of $\mathbb{C}$ s.t. $\exists \omega \in \mathcal{M}$ for which $E(\omega) = 0$ and $f(\omega^c) \subset R_E(f)$.
(i.e. $R_E(f)$ contains $f(\lambda)$ for "almost every" $\lambda \in \mathfrak{X}$.)

Proof "a)" Let $f:\Xi\to\mathbb{C}$ be measurable. Let $I\subset\mathbb{C}\times\mathbb{N}$ collect those $z\in\mathbb{C}$, $n\in\mathbb{N}$ for which $\mathrm{Re}\,z, \mathrm{Im}\,z\in\mathbb{Q}$ and $E(f^{\leftarrow}B(z,\frac{1}{n}))=0$. $\Rightarrow R_E(f)=\bigcap_{(z,n)\in I} B(z,\frac{1}{n})^c$

$\Rightarrow R_E(f)^c=\bigcup_{(z,n)\in I} B(z,\frac{1}{n})$ is open in $\mathbb{C}$. In addition, $I$ $\Rightarrow R_E(f)$ is closed.
is countable, and thus by 19.2. f), $0=E(\bigcup_{(z,n)\in I} f^{\leftarrow}B(z,\frac{1}{n}))$
$=E(f^{\leftarrow}(\bigcup_I B(z,\frac{1}{n})))=E(f^{\leftarrow}(R_E(f)^c))$. Thus $R_E(f)=\emptyset \Rightarrow 0=E(f^{\leftarrow}\mathbb{C})=E(\Xi)=1$ ↯.
Set $\omega:=f^{\leftarrow}(R_E(f)^c)\in\mathcal{M}$. $\Rightarrow E(\omega)=0$ and $\omega^c=\Xi\setminus\omega$
$=(f^{\leftarrow}(R_E(f)^c))^c=f^{\leftarrow}(R_E(f))$, thus $\lambda\in\omega^c\Rightarrow f(\lambda)\in R_E(f)$.
Suppose then that $R'\subset\mathbb{C}$ is closed and $\omega\in\mathcal{M}$ is s.t. $E(\omega)=0$ and $f(\omega^c)\subset R'$. Since $R'^c$ is open $\exists I'\subset\mathbb{C}\times\mathbb{N}$, with $(z,n)\in I'\Rightarrow \mathrm{Re}\,z,\mathrm{Im}\,z\in\mathbb{Q}$ s.t. $R'^c=\bigcup_{(z,n)\in I'} B(z,\frac{1}{n})$. Suppose $(z,n)\in I'$

$\Rightarrow U:=B(z,\frac{1}{n})\subset R'^c\subset f(\omega^c)^c$. Thus if $\lambda\in f^{\leftarrow}U\Rightarrow f(\lambda)\in U$
$\Rightarrow \lambda\notin\omega^c\Rightarrow\lambda\in\omega$. Therefore, $f^{\leftarrow}U\subset\omega$, and as $U$ is open $f^{\leftarrow}U\in\mathcal{M}$. Thus $\forall\psi\in\mathcal{H}$: $\|E(f^{\leftarrow}U)\psi\|^2=(\psi,E(f^{\leftarrow}U)\psi)$
$=\int E_{\psi,\psi}(d\lambda)\,\mathbb{1}(\lambda\in f^{\leftarrow}U)\le\int E_{\psi,\psi}(d\lambda)\,\mathbb{1}(\lambda\in\omega)=(\psi,E(\omega)\psi)=0$.
$\therefore E(f^{\leftarrow}U)=0\Rightarrow(z,n)\in I$ by def. of $R_E(f)$. Thus $I'\subset I$
$\Rightarrow R'^c\subset R_E(f)^c\Rightarrow R_E(f)\subset R'$. $\therefore$ $R_E(f)$ is minimal.
Since $f(\Xi)\subset\overline{f(\Xi)}$ and $E(\Xi^c)=E(\emptyset)=0$ $\Rightarrow R_E(f)\subset\overline{f(\Xi)}$ □

"b)" If $f,g$ are measurable $\Rightarrow f+g$, $fg$, and $\alpha f$, $\alpha\in\mathbb{C}$ are all measurable [RCA, 1.9.]. In addition, $|f(\lambda)+g(\lambda)|\le\|f\|_\infty+\|g\|_\infty$, $|(fg)(\lambda)|=|f(\lambda)||g(\lambda)|\le\|f\|_\infty\|g\|_\infty$, and $\|\alpha f\|_\infty=|\alpha|\,\|f\|_\infty$.
Thus $L^\infty_{pre}(E)$ is a complex vector space with multip. $(f,g)\mapsto fg$.
It is obvious that it is a commut. complex algebra (e.g. $(f(gh))(\lambda)$ $:=f(\lambda)(gh)(\lambda)=f(\lambda)(g(\lambda)h(\lambda))=((fg)h)(\lambda)$, etc.) with unit
(*) $e(\lambda)=1\ \forall\lambda\in\Xi$. (since $e^{\leftarrow}U=\begin{cases}\emptyset, & 1\notin U\\ \Xi, & 1\in U\end{cases}$, $e$ is measurable.)
$\Rightarrow\|e\|_\infty=1$. Also $\|fg\|_\infty\le\|f\|_\infty\|g\|_\infty$, as seen above.
Now if $f_n$, $n\in\mathbb{N}_+$, is Cauchy in $L^\infty_{pre}(E)\Rightarrow\|f_n-f_m\|_\infty\xrightarrow{n,m\to\infty}0$
$\Rightarrow\forall\lambda\in\Xi$: $|f_n(\lambda)-f_m(\lambda)|\xrightarrow{n,m\to\infty}0\Rightarrow\forall\lambda\in\Xi$; $f_n(\lambda)$ is Cauchy in $\mathbb{C}$
$\Rightarrow\forall\lambda\in\Xi\ \exists f(\lambda)=\lim_{n\to\infty}f_n(\lambda)\Rightarrow|f(\lambda)|=\lim_{n\to\infty}|f_n(\lambda)|$
$\le\sup_{n\in\mathbb{N}_+}\|f_n\|_\infty<\infty$ since $\exists n_0\in\mathbb{N}_+$ s.t. $\|f_n-f_m\|_\infty\le1\ \forall n,m\ge n_0$.
$\Rightarrow\|f_n\|_\infty\le\|f_n-f_{n_0}\|_\infty+\|f\|_{n_0}\le1+\|f\|_{n_0}\ \forall n\ge n_0$.
Thus $\lambda\mapsto f(\lambda)$ is a map $\Xi\to\mathbb{C}$ for which $\|f\|_\infty<\infty$.
Since $f_n\to f$ pointwise and each $f_n$ is measurable, by [RCA, 1.14.] $\Rightarrow f$ is measurable $\Rightarrow f\in L^\infty_{pre}(E)$. Now if $\varepsilon>0$, then

(*) $\|f\|_\infty=0\Rightarrow f(\lambda)=0\ \forall\lambda\Rightarrow f=0\in L^\infty_{pre}$, as $0^{\leftarrow}U=\begin{cases}\emptyset, & 0\notin U\\ \Xi, & 0\in U\end{cases}$. Thus $\|\cdot\|_\infty$ is a norm.

$\exists n_0$ s.t. $\|f_n - f_m\|_\infty < \frac{\varepsilon}{3}$ $\forall n,m \geq n_0$. Thus if $n \geq n_0$ we have $|f(\lambda) - f_n(\lambda)| \leq |f(\lambda) - f_m(\lambda)| + |f_m(\lambda) - f_n(\lambda)|$
$< |f(\lambda) - f_m(\lambda)| + \frac{\varepsilon}{3}$ $\forall m \geq n_0$ $\Rightarrow$ $|f(\lambda) - f_n(\lambda)| < 2\frac{\varepsilon}{3}$ $\forall n \geq n_0$
since $\exists m \geq n_0$ s.t. $|f(\lambda) - f_m(\lambda)| < \frac{\varepsilon}{3}$. $\Rightarrow \|f - f_n\|_\infty < \varepsilon$ $\forall n \geq n_0$
$\Rightarrow f_n \to f$ in norm. $\therefore$ $\breve{L}_{pre}(E)$ is a commutative Banach algebra.

"c)" Now $f \in N \Rightarrow R_E(f) = \{0\}$ (as $R_E(f) \neq \emptyset$) $\Rightarrow \exists \omega \in M$ s.t. $E(\omega) = 0$ and $f(\lambda) = 0$ $\forall \lambda \notin \omega$. If $g \in N$, $\exists \omega' \in M$ s.t. $E(\omega') = 0$ and $g(\lambda) = 0$ $\forall \lambda \notin \omega'$. Thus if $\omega_0 = \omega \cup \omega' \in M$ we have $E(\omega_0) = 0$ (by 19.2.f) and $g(\lambda) = 0 = f(\lambda)$ $\forall \lambda \notin \omega_0$
$\Rightarrow$ $\alpha f(\lambda) + \beta g(\lambda) = 0$ $\forall \alpha, \beta \in \mathbb{C}$, $\lambda \in \omega_0^c$. Since $\{0\}$ is closed, "a)" implies that $R_E(\alpha f + \beta g) \subset \{0\} \Rightarrow \|\alpha f + \beta g\|_\infty = 0$
$\Rightarrow \alpha f + \beta g \in N$. If $f \equiv \alpha \in \mathbb{C}$, then by "a)" $\emptyset \neq R_E(f) \subset \overline{\{\alpha\}} = \{\alpha\}$
$\Rightarrow R_E(f) = \{\alpha\}$. Thus $0 \in N$ but $1 \notin N$ $\Rightarrow$ $N$ is a proper subspace. If $\tilde{g} \in L^\infty_{pre}(E) \Rightarrow \forall \lambda \notin \omega$ we have $(\tilde{g}f)(\lambda) = \tilde{g}(\lambda) f(\lambda) = 0$ and since $E(\omega) = 0 \Rightarrow R_E(\tilde{g}f) \subset \{0\}$
$\Rightarrow \tilde{g}f \in N$. $\therefore$ $N$ is a proper ideal.

Finally, if $f \in \overline{N} \Rightarrow \exists f_n \in N$ s.t. $f_n \to f$ in norm.
$\Rightarrow \exists \omega_n \in M$ s.t. $E(\omega_n) = 0$ and $f_n(\lambda) = 0$ $\forall \lambda \notin \omega_n$.
Set $\omega := \bigcup_{n=1}^\infty \omega_n \overset{19.2.f)}{\Longrightarrow} E(\omega) = 0$ and $\omega^c = \bigcap_{n=1}^\infty \omega_n^c$. Thus if $\lambda \in \omega^c$ and $n \in \mathbb{N}_+$ $\Rightarrow \lambda \in \omega_n^c \Rightarrow f_n(\lambda) = 0$. Therefore, $\forall \lambda \in \omega^c$: $f(\lambda) = \lim_{n \to \infty} f_n(\lambda) = 0$. $\therefore$ $R_E(f) \subset \{0\} \Rightarrow f \in N$.
$\therefore$ $N$ is closed. $\square$

<u>19.8. Corollary:</u> Define $L^\infty(E) := L^\infty_{pre}(E)/N$. Then

a) $L^\infty(E)$ is a commutative Banach algebra.
b) For any $f \in L^\infty_{pre}(E)$, $\|\pi(f)\| = \|f\|_{E,\infty}$ and $\sigma(\pi(f)) = R_E(f)$ where $\pi : L^\infty_{pre}(E) \to L^\infty(E)$ denotes the quotient map.

<u>Proof:</u> Denote $\mathcal{C} := L^\infty_{pre}(E)$ and $L^\infty := L^\infty(E)$. "a)" follows from from 19.7. applying 17.4.b). Suppose $f \in \mathcal{C}$ and $g \in N$.
$\Rightarrow \exists \omega_0, \omega \in M$ s.t. $E(\omega_0) = 0 = E(\omega)$ and $f(\lambda) \in R_E(f)$ $\forall \lambda \notin \omega$, $g(\lambda) = 0$ $\forall \lambda \notin \omega_0 \Rightarrow E(\omega \cup \omega_0) = 0$ and $f(\lambda) - g(\lambda) = f(\lambda) \in R_E(f)$ $\forall \lambda \notin \omega \cup \omega_0$. Since $R_E(f)$ is closed $\overset{19.7.a)}{\Longrightarrow}$ $R_E(f-g) \subset R_E(f)$
$\Rightarrow R_E(f) = R_E(f-g-(-g)) \subset R_E(f-g)$ since $-g \in N$. Thus $R_E(f-g) = R_E(f)$.
By 19.7. a) $\Rightarrow$ $R_E(f) \subset \overline{f(X)}$ $\overset{(*)}{\Rightarrow}$ $\|f\|_{E,\infty} \leq \|f\|_\infty < \infty$ $\forall f \in \mathcal{C}$.

(*) $\varepsilon > 0 \Rightarrow \exists z_\varepsilon \in R_E(f)$ s.t. $|z_\varepsilon| + \varepsilon > \|f\|_{E,\infty} \Rightarrow \exists \lambda_\varepsilon \in X$ s.t. $|f(\lambda_\varepsilon) - z_\varepsilon| < \varepsilon \Rightarrow \|f\|_\infty \geq |f(\lambda_\varepsilon)| \geq |z_\varepsilon| - |f(\lambda_\varepsilon) - z_\varepsilon| > \|f\|_{E,\infty} - 2\varepsilon$.

Since $\|\pi(f)\| = \inf\{\|f-g\|_\infty \mid g\in N\} \Rightarrow \exists g_n \in N$ s.t. $\|f-g_n\|_\infty \to \|\pi(f)\| \Rightarrow R_E(f) = R_E(f-g_n) \Rightarrow \|f\|_{E,\infty} = \|f-g_n\|_{E,\infty} \le \|f-g_n\|_\infty \Rightarrow \|f\|_{E,\infty} \le \|\pi(f)\|$. But by 19.7 a) $\exists\, \omega\in \mathfrak{M}$ s.t. $E(\omega)=0$ and $f(\omega^c)\subset R_E(f)$. Define $g(\lambda) = \mathbb{1}(\lambda\in\omega) f(\lambda)$

$\Rightarrow g\in N$ since $\lambda \mapsto \mathbb{1}(\lambda\in\omega)\in N \Rightarrow f(\lambda)-g(\lambda) = \begin{cases} \lambda\in\omega: 0 \\ \lambda\notin\omega: f(\lambda)\in R_E(f)\end{cases}$

$\Rightarrow |f(\lambda)-g(\lambda)| \le \|f\|_{E,\infty}\ \forall\lambda\in X$

$\Rightarrow \|\pi(f)\| \le \|f-g\|_\infty \le \|f\|_{E,\infty}$. Thus $\|\pi(f)\| = \|f\|_{E,\infty}$.

Suppose then $z_0\notin R_E(f)$. Since $R_E(f)\neq\emptyset$, can choose $z_1\in R_E(f)$. By 19.7 a) $\exists\,\omega\in \mathfrak{M}$ s.t. $E(\omega)=0$ and $f(\lambda)\in R_E(f)\ \forall\lambda\notin\omega$. Define $g(\lambda) := \mathbb{1}(\lambda\in\omega)(z_1 - f(\lambda))$

$\Rightarrow g\in N$, as above ($z_1 1 - f\in\mathcal{C}$) $\Rightarrow \pi(f+g)=\pi(f)$

and $(f+g)(\lambda) = \begin{cases}\lambda\notin\omega: f(\lambda)\\ \lambda\in\omega: \cancel{f(\lambda)}+z_1-\cancel{f(\lambda)}\end{cases} \in R_E(f)\ \forall\lambda\in X$.

Since $R_E(f)$ is closed in $\mathbb{C}$ $\Rightarrow \exists\,\varepsilon>0$ s.t. $B(z_0,\varepsilon)\subset R_E(f)^c$

$\Rightarrow \forall z\in R_E(f): |z-z_0|\ge\varepsilon \Rightarrow |z_0-(f+g)(\lambda)|\ge\varepsilon\ \forall\lambda\in X$

$\Rightarrow z_0 1-(f+g)\in G(\mathcal{C})$ (the inverse is $\lambda\mapsto \frac{1}{z_0-f(\lambda)-g(\lambda)}$ which is meas. as $z\mapsto z^{-1}$ is contin. on $\mathbb{C}\setminus\{0\}$.)

$\Rightarrow z_0\pi(1)-\pi(f+g) = z_0\pi(1)-\pi(f)\in G(L^\infty)$. (as $\pi$ is homom.)

$\Rightarrow z_0\notin\sigma(\pi(f))$.

Finally, assume $z_0\notin\sigma(\pi(f)) \Rightarrow z_0\pi(1)-\pi(f)\in G(L^\infty)$

$\overset{\pi \text{ homom.}}{\Rightarrow} \exists f_1\in\mathcal{C}$ s.t. $\pi(f_1)\pi(z_0 1-f)=\pi(1)$

$\Rightarrow f_1(z_0 1-f)-1\in N \Rightarrow \exists\,\omega\in \mathfrak{M}$ s.t. $E(\omega)=0$ and $f_1(\lambda)(z_0-f(\lambda))=1\ \forall\lambda\notin\omega$

$\Rightarrow \left|\frac{1}{z_0-f(\lambda)}\right| = |f_1(\lambda)|\le\|f_1\|_\infty =: M_1\ \forall\lambda\notin\omega$

$\Rightarrow |z-f(\lambda)|\ge|z_0-f(\lambda)|-|z-z_0|\ge\frac{1}{2M_1}>0\ \forall\ |z-z_0|<\frac{1}{2M_1}$

$\Rightarrow f(\lambda)\in B_0^c\ \forall\lambda\notin\omega$ where $B_0 := B(z_0,\frac{1}{2M_1})$ is open

$\Rightarrow B_0^c$ is closed, and as $E(\omega)=0$

$\overset{19.7.a)}{\Rightarrow} R_E(f)\subset B_0^c \Rightarrow z_0\in B_0\subset R_E(f)^c \Rightarrow z_0\notin R_E(f)$ □

* As is typical in measure theory, usually the distinction between $f\in L^\infty_{pre}$ and $\pi(f)\in L^\infty$ is ignored in the notation.

## 19.9 Theorem (symbolic calculus)

Suppose $\mathcal{M}$ is a $\sigma$-algebra in a set $\mathfrak{X}$, $\mathcal{H}$ is a Hilbert space, and $E: \mathcal{M} \to \mathcal{B}(\mathcal{H})$ is a resolution of the identity.
Then the formula

$$(\varphi, \Psi(f)\psi) = \int_{\mathfrak{X}} E_{\varphi,\psi}(d\lambda)\, f(\lambda) \quad \forall \varphi, \psi \in \mathcal{H}$$

defines an isometric isomorphism $\Psi: L^\infty(E) \to \mathcal{A}$ where $\mathcal{A}$ is a closed normal subalgebra of $\mathcal{B}(\mathcal{H})$.
In addition,

a) $\Psi(f^*) = \Psi(f)^* \quad \forall f \in L^\infty(E)$
b) $\|\Psi(f)\varphi\|^2 = \int_{\mathfrak{X}} E_{\varphi,\varphi}(d\lambda)\, |f(\lambda)|^2 \quad \forall f \in L^\infty(E), \varphi \in \mathcal{H}$
d) If $S \in \mathcal{B}(\mathcal{H})$, then $SE(\omega) = E(\omega)S \;\; \forall \omega \in \mathcal{M}$ iff
$S\Psi(f) = \Psi(f)S \quad \forall f \in L^\infty(E)$.
c) $\sigma(\Psi(f)) = R_E(f) \quad \forall f \in L^\infty(E)$. $\quad (\sigma = \sigma_{\mathcal{B}(\mathcal{H})})$

Proof: In measure theory, a simple function is a measurable function whose range consists of finitely many points.
For any simple $s: \mathfrak{X} \to \mathbb{C}$, let $N := |s(\mathfrak{X})|$, $\alpha_i \in \mathbb{C}$, $i = 1, \dots, N$, enumerate the points on $s(\mathfrak{X})$, and define $\omega_i := s^{\leftarrow}(\{\alpha_i\})$, $i = 1, \dots, N$. Since $s$ is measurable and $\omega_i = s^{\leftarrow}(B(\alpha_i, \varepsilon))$ for some $\varepsilon > 0$ $\Rightarrow \omega_i \in \mathcal{M}$. In addition, $\omega_i$ are disjoint, $\bigcup_{i=1}^{N} \omega_i = \mathfrak{X}$, and $s(\lambda) = \sum_{i=1}^{N} \alpha_i \mathbb{1}(\lambda \in \omega_i)$ $\forall \lambda \in \mathfrak{X}$, and as $s$ is bounded $s \in L^\infty_{pre}(E)$. Define then
$\tilde{s} := \sum_{i=1}^{N} \alpha_i E(\omega_i) \in \mathcal{B}(\mathcal{H})$. (Note that $\tilde{s}$ is uniquely determined by $s$.) Since every $E(\omega)$ is self-adjoint $\Rightarrow \tilde{s}^* = \sum_{i=1}^{N} \alpha_i^* E(\omega_i)$
$= \widetilde{(s^*)}$.

Consider then two simple functions $s, s'$ and define $\alpha_i, \alpha_i', \omega_i, \omega_i', N, N'$ as above. Then $\forall \lambda \in \mathfrak{X}$:

$$(ss')(\lambda) = s(\lambda)s'(\lambda) = \sum_{i=1}^{N}\sum_{i'=1}^{N'} \alpha_i \alpha_{i'}' \underbrace{\mathbb{1}(\lambda \in \omega_i)\, \mathbb{1}(\lambda \in \omega_{i'}')}_{= \mathbb{1}(\lambda \in \omega_i \cap \omega_{i'}')}$$

Thus $s'' := ss'$ is also simple with $\alpha_j''$ enumerating points in $\{\alpha_i \alpha_{i'}'\}_{i,i'}$ and $\omega_j'' := \bigcup_{(i,i') \in I_j} (\omega_i \cap \omega_{i'}')$ where
(note: $\omega_i \cap \omega_{i'}'$ are disjoint)

$I_j := \{(i,i') \mid \alpha_i \alpha_{i'}' = \alpha_j''\}$ and $N'' \le NN' < \infty$.

$$\Rightarrow \widetilde{(s'')} = \sum_{j=1}^{N''} \alpha_j'' E(\omega_j'') = \sum_{j=1}^{N''} \alpha_j'' \sum_{(i,i') \in I_j} E(\omega_i \cap \omega_{i'}')$$

$$= \sum_{i,i'} \underbrace{\sum_{j=1}^{N''} \alpha_j'' \mathbb{1}((i,i') \in I_j)}_{= \alpha_i \alpha_{i'}' \mathbb{1}((i,i') \in I_j)} E(\omega_i) E(\omega_{i'}')$$

$= \left(\sum_{i=1}^{N} \alpha_i E(\omega_i)\right)\left(\sum_{i'=1}^{N'} \alpha_{i'}' E(\omega_{i'}')\right)$, since $\sum_{j=1}^{N''} \mathbb{1}((i,i') \in I_j) = 1 \quad \forall i, i'$.

$\Rightarrow \widetilde{(s'')} = \tilde{s}\,\tilde{s'}$. If $\beta, \beta' \in \mathbb{C}$ and $s'' := \beta s + \beta' s'$

$\Rightarrow s''(\lambda) = \sum_{i=1}^{N} \beta\alpha_i \mathbb{1}(\lambda \in \omega_i) + \sum_{i'=1}^{N'} \beta'\alpha_{i'}' \mathbb{1}(\lambda \in \omega_{i'}')$

$= \sum_{i',i} (\beta\alpha_i + \beta'\alpha_{i'}')\mathbb{1}(\lambda \in \omega_i \cap \omega_{i'}')$, since $\omega_i = \bigcup_{i'=1}^{N'} (\omega_i \cap \omega_{i'}')$ and $\omega_{i'}' = \bigcup_{i=1}^{N} (\omega_{i'}' \cap \omega_i)$.

$\Rightarrow \mathbb{1}(\lambda \in \omega_i) = \sum_{i'=1}^{N'} \mathbb{1}(\lambda \in \omega_i \cap \omega_{i'}')$ and $\mathbb{1}(\lambda \in \omega_{i'}') = \sum_{i=1}^{N} \mathbb{1}(\lambda \in \omega_i \cap \omega_{i'}')$

There $s''$ is simple, and if $\alpha_j''$ enumerates $\{\beta\alpha_i + \beta'\alpha_{i'}'\}$ then $I_j := \{(i,i') \mid \beta\alpha_i + \beta'\alpha_{i'}' = \alpha_j''\} \Rightarrow \omega_j'' = \bigcup_{(i,i') \in I_j} (\omega_i \cap \omega_{i'}')$. and $N'' \le NN' < \infty$. Thus now

$$\widetilde{(s'')} = \sum_{j=1}^{N''} \alpha_j'' E(\omega_j'') = \sum_{i,i'} \sum_{j=1}^{N''} (\beta\alpha_i + \beta'\alpha_{i'}')\underbrace{\mathbb{1}((i,i') \in I_j)}_{E(\omega_i \cap \omega_{i'}')}$$

$$= \sum_{i=1}^{N} \sum_{i'=1}^{N'} \beta\alpha_i E(\omega_i \cap \omega_{i'}') + \sum_{i'=1}^{N'} \sum_{i=1}^{N} \beta'\alpha_{i'}' E(\omega_i \cap \omega_{i'}')$$

$$\overset{\text{E finitely addit.}}{=} \beta \sum_{i=1}^{N} \alpha_i E(\underbrace{\bigcup_{i'=1}^{N'}(\omega_i \cap \omega_{i'}')}_{=\omega_i}) + \beta' \sum_{i'=1}^{N'} \alpha_{i'}' E(\underbrace{\bigcup_{i=1}^{N}(\omega_i \cap \omega_{i'}')}_{=\omega_{i'}'})$$

$$= \beta\tilde{s} + \beta'\widetilde{(s')}.$$

Now for any simple $s$, and $\varphi, \psi \in \mathcal{H}$ we have

$(\varphi, \tilde{s}\psi) = \sum_{i=1}^{N} \alpha_i (\varphi, E(\omega_i)\psi) = \sum_{i=1}^{N} \alpha_i E_{\varphi,\psi}(\omega_i)$

$$= \sum_{i=1}^{N} \alpha_i \int_{\underline{X}} E_{\varphi,\psi}(d\lambda)\mathbb{1}(\lambda \in \omega_i) = \int_{\underline{X}} E_{\varphi,\psi}(d\lambda)\, s(\lambda).$$

On the other hand, by the above results,

$\tilde{s}^*\tilde{s} = \widetilde{(s^*)}\tilde{s} = \widetilde{(s^*s)} = \widetilde{|s|^2} \Rightarrow \forall \psi \in \mathcal{H}$

$\|\tilde{s}\psi\|^2 = (\psi, \tilde{s}^*\tilde{s}\psi) = (\psi, \widetilde{|s|^2}\psi) = \int_{\underline{X}} E_{\psi,\psi}(d\lambda)\,|s(\lambda)|^2$.

Let then $I_0 := \{i \in \{1,..,N\} \mid E(\omega_i) = 0\}$, and define $s_0 := \sum_{i \in I_0} \alpha_i \mathbb{1}(\lambda \in \omega_i) \Rightarrow s_0$ is simple. Also $s_0(\lambda) = 0$ $\forall \lambda \notin \bigcup_{i \in I_0} \omega_i$ where $E(\bigcup_{i \in I_0} \omega_i) = 0$

$\Rightarrow s_0 \in N$. But $\tilde{s}_0 = \sum_{i \in I_0} \alpha_i E(\omega_i) = 0 \Rightarrow \tilde{s} = \widetilde{(s - s_0)}$

Since $1 = E(\underline{X}) = \sum_{i=1}^{N} E(\omega_i) \Rightarrow I_0 \ne \{1,..,N\} =: I$

$\Rightarrow \exists j \in I \setminus I_0$ s.t. $|\alpha_i| \le |\alpha_j| \quad \forall i \in I \setminus I_0$

Since $E(\omega_j) \ne 0 \Rightarrow \exists \psi_0 \in R(E(\omega_j))$, $\psi_0 \ne 0$,

$\Rightarrow \forall \varphi \in \mathcal{H}: (\varphi, \tilde{s}\psi_0) = \sum_{i=1}^{N} \alpha_i (\underbrace{E(\omega_i)\varphi}_{\in R(E(\omega_i))}, \psi_0)$

$\overset{19.2.c)}{=} \alpha_j(\varphi, E(\omega_j)\psi_0) = (\varphi, \alpha_j\psi_0) \Rightarrow \tilde{s}\psi_0 = \alpha_j\psi_0$

$\Rightarrow \| \tilde{S} \frac{\psi_0}{\|\psi_0\|} \| = \frac{1}{\|\psi_0\|} \| \alpha_j \psi_0 \| = |\alpha_j|$ where $\| \frac{\psi_0}{\|\psi_0\|} \| = 1$.

$\Rightarrow \|\tilde{S}\| \geq |\alpha_j|$. On the other hand, $\forall \psi \in \mathcal{H}$
$\| \tilde{S}\psi \|^2 = \| (\widetilde{S - S_0})\psi \|^2 = \int_{\Xi} E_{\psi,\psi}(d\lambda) \underbrace{|S(\lambda) - S_0(\lambda)|^2}_{= \sum_{i \in I \setminus I_0} \alpha_i \mathbb{1}(\lambda \in \omega_i)}$

$\leq |\alpha_j|^2 \int_{\Xi} E_{\psi,\psi}(d\lambda) \overset{19.7a)}{=} |\alpha_j|^2 \|\psi\|^2$

$\Rightarrow \|\tilde{S}\| \leq |\alpha_j|$. Thus $\|\tilde{S}\| = |\alpha_j|$. Suppose $i \in I$. then $\exists z \in \mathbb{Q} + i\mathbb{Q}$, $n \in \mathbb{N}_+$ s.t. $\alpha_i \in B(z, \frac{1}{n})$ but $\alpha_{i'} \notin B(z, \frac{1}{n})$ $\forall i' \neq i$. $\Rightarrow S^{\leftarrow} B(z, \frac{1}{n}) = S^{\leftarrow}\{\alpha_i\} = \omega_i$.
Thus if $i \in I_0 \Rightarrow E(S^{\leftarrow} B(z, \frac{1}{n})) = 0 \Rightarrow \alpha_i \in B(z, \frac{1}{n}) \subset R_E(f)^c$
$\Rightarrow \alpha_i \notin R_E(f)$. If $i \in I \setminus I_0$ and $\alpha_i \in B(z, \varepsilon)$ for some $z \in \mathbb{C}, \varepsilon > 0$
$\Rightarrow \omega_i = S^{\leftarrow}\{\alpha_i\} \subset S^{\leftarrow} B(z, \varepsilon) =: \omega \Rightarrow E(\omega) = E(\omega_i) + E(\omega \setminus \omega_i)$,
Since $E(\omega_i) \neq 0 \Rightarrow \exists \psi_0 \neq 0$, $\psi_0 \in R(E(\omega_i))$
$\Rightarrow \forall \varphi \in \mathcal{H}: (\varphi, E(\omega \setminus \omega_i)\psi_0) = (E(\omega \setminus \omega_i)\varphi, \psi_0) = 0$
$\Rightarrow E(\omega \setminus \omega_i)\psi_0 = 0 \Rightarrow E(\omega)\psi_0 = E(\omega_i)\psi_0 = \psi_0 \neq 0$
$\Rightarrow E(\omega) \neq 0$. Therefore, (by def.) $\alpha_i \in R_E(S)$. Since $R_E(S) \subset \{\alpha_i\}_{i \in I} = \{\alpha_i\}_{i \in I}$ we can conclude that $R_E(S) = \{\alpha_i\}_{i \in I \setminus I_0} \Rightarrow \|S\|_{E,\infty} = |\alpha_j|$.

$\therefore \|\tilde{S}\| = \|S\|_{E,\infty} \overset{19.6.}{=} \|\pi(S)\|$ $\forall S =$ simple.

Suppose then that $f \in L^\infty_{pre}(E)$. Since $f$ is bounded and measurable, it can approximated uniformly by a sequence of simple functions; i.e., $\exists S_n \in L^\infty_{pre}(E)$, $n \in \mathbb{N}_+$, s.t. each $S_n$ is simple and $\|f - S_n\|_\infty \to 0$ for $n \to \infty$.
(For bounded positive $f$, the construction of $S_n$ is the same as in the proof of 1.17. in [RCA]. Since any bounded complex $f$ can be written as a lin. combinat. of four bounded positive functions, the general case follows.) But then also
$\|\pi(f) - \pi(S_n)\| = \|\pi(f - S_n)\| \leq \|f - S_n\|_\infty \overset{n\to\infty}{\to} 0 \Rightarrow \pi(S_n) \to \pi(f)$.
$\Rightarrow \|\tilde{S}_n - \tilde{S}_m\| = \|(\widetilde{S_n - S_m})\| = \|\pi(S_n - S_m)\| = \|\pi(S_n) - \pi(S_m)\|$
$\to 0$ for $n, m \to \infty$ (since $\pi(S_n)$ is Cauchy)
$\Rightarrow \tilde{S}_n$ is Cauchy in $\mathcal{B}(\mathcal{H}) \Rightarrow \exists \tilde{f} = \lim_{n\to\infty} \tilde{S}_n$ in $\mathcal{B}(\mathcal{H})$.
Suppose then that $S_n' \to f$ is another sequence of this type.
$\Rightarrow \tilde{S}_n' \to T$ in $\mathcal{B}(\mathcal{H}) \Rightarrow \|T - \tilde{f}\| = \lim_{n\to\infty} \|\tilde{S}_n' - \tilde{S}_n\| = \lim \|(\widetilde{S_n' - S_n})\|$
$= \lim_{n\to\infty} \|\pi(S_n' - S_n)\|$. As $\|\pi(S_n' - S_n)\| \leq \|S_n' - S_n\|_\infty \leq \|S_n' - f\|_\infty + \|f - S_n\|_\infty$
$\to 0$ as $n \to \infty \Rightarrow T = \tilde{f}$. Thus the limit is uniquely determined by $f$. (it does not depend on the choice of $(S_n)$.)

(NB) This proves that the RHS of (*) does not depend on the choice of representative $\Rightarrow$ (*) determines $\Psi(f)$ uniquely.

Finally, if $f' \sim f \Rightarrow g := f' - f \in N$. Choose sequence of simple functions, $s_n, \delta_n, n \in \mathbb{N}_+$, s.t. $\|f - s_n\|_\infty \to 0$ and $\|g - \delta_n\|_\infty \to 0 \Rightarrow \|\tilde{\delta}_n\| = \|\pi(\delta_n)\| = \|\pi(\delta_n - g)\| \le \|\delta_n - g\|_\infty \to 0 \Rightarrow \tilde{\delta}_n \to 0$ as $n \to \infty$. But now $\|f' - (s_n + \delta_n)\|_\infty \le \|f - s_n\|_\infty + \|g - \delta_n\|_\infty \to 0$, and thus by the above results $\tilde{f'} = \lim_{n\to\infty} \widetilde{(s_n + \delta_n)} = \lim_{n\to\infty} (\tilde{s}_n + \tilde{\delta}_n) = \tilde{f} + 0 = \tilde{f}$. This

shows that $\Psi(\pi(f)) := \tilde{f}$ $\forall f \in L^\infty_{pre}(E)$ defines a map $\Psi: L^\infty(E) \to \mathcal{B}(\mathcal{H})$.

Consider then $f, f' \in L^\infty_{pre}(E)$, with simple approximations $s_n \to f$ and $s_n' \to f'$. Then $\forall \beta, \beta' \in \mathbb{C}$, $\beta\tilde{f} + \beta'\tilde{f'}$ $= \lim_{n\to\infty} (\beta\tilde{s}_n + \beta'\tilde{s}_n') = \lim_{n\to\infty} \widetilde{(\beta s_n + \beta' s_n')} = \widetilde{(\beta f + \beta' f')}$

since $\|\beta f + \beta' f' - (\beta s_n + \beta' s_n')\|_\infty \le |\beta| \|f - s_n\|_\infty + |\beta'| \|f' - s_n'\|_\infty \to 0$ as $n \to \infty$. Thus $\Psi(\beta\pi(f) + \beta'\pi(f')) = \Psi(\pi(\beta f + \beta' f'))$ $= \beta\Psi(\pi(f)) + \beta'\Psi(\pi(f'))$, since $\pi$ is homom. (17.4.b).

As $\|f'f - s_n' s_n\|_\infty \le \|f'\|_\infty \|f - s_n\|_\infty + \|f\|_\infty \|f' - s_n'\|_\infty + \|f - s_n\|_\infty \|f' - s_n'\|_\infty \to 0$ as $n \to \infty \Rightarrow \widetilde{f'f} = \lim_{n\to\infty} \widetilde{(s_n' s_n)} = \lim_{n\to\infty} (\tilde{s}_n' \tilde{s}_n)$ $= \tilde{f'}\tilde{f}$ by 16.2. Thus

$\Psi(\pi(f')\pi(f)) = \Psi(\pi(f'f)) = \Psi(\pi(f'))\Psi(\pi(f))$.

$\therefore$ $\Psi$ is a homomorphism. Also $\|\Psi(\pi(f))\| = \|\tilde{f}\| = \lim_{n\to\infty} \|\tilde{s}_n\|$ $= \lim_{n\to\infty} \|\pi(s_n)\| = \|\pi(f)\|$ since $\pi(s_n) \to \pi(f)$ in $L^\infty(E)$.

Thus $\Psi$ is an isometry. If $\Psi(\pi(f)) = \Psi(\pi(f'))$ $\Rightarrow 0 = \Psi(\pi(f - f')) \Rightarrow 0 = \|0\| = \|\pi(f - f')\| \Rightarrow \pi(f) - \pi(f') = 0$ $\Rightarrow \pi(f) = \pi(f')$. Thus $\Psi$ is an isometric isomorphism $L^\infty(E) \to \mathcal{A} := \Psi(L^\infty(E)) \subset \mathcal{B}(\mathcal{H})$. $\Rightarrow \mathcal{A}$ is a commutative subalgebra of $\mathcal{B}(\mathcal{H})$.

Finally, since $\mathcal{B}(\mathcal{H})$ is a $C^*$-algebra and $\|f^* - s_n^*\|_\infty$ $= \|f - s_n\|_\infty \to 0$, we have $\tilde{f}^* = \lim_{n\to\infty} \tilde{s}_n^* = \lim_{n\to\infty} \widetilde{(s_n^*)} = \widetilde{f^*} \in \mathcal{A}$. This implies "a)". Also if $\varphi \in \mathcal{H}$,

$\|\Psi(\pi(f))\varphi\|^2 = \|\tilde{f}\varphi\|^2 = \lim_{n\to\infty} \|\tilde{s}_n\varphi\|^2 = \lim_{n\to\infty} \left( \int_{\mathbb{X}} E_{\varphi,\varphi}(d\lambda) |s_n(\lambda)|^2 \right)$

$\overset{DCT}{=} \int_{\mathbb{X}} E_{\varphi,\varphi}(d\lambda) |f(\lambda)|^2$. Thus "b)" holds. Similarly, $(\varphi, \tilde{f}\psi)$ $= \lim_{n\to\infty} (\varphi, \tilde{s}_n\psi) = \lim \int_{\mathbb{X}} E_{\varphi,\psi}(d\lambda) s_n(\lambda) \overset{DCT}{=} \int_{\mathbb{X}} E_{\varphi,\psi}(d\lambda) f(\lambda) \Rightarrow (*)$. (NB)

If $T \in \bar{\mathcal{A}} \Rightarrow \exists f_n \in L^\infty_{pre}(E)$ s.t. $\tilde{f}_n \to T$ in norm. $\Rightarrow \|\tilde{f}_n - \tilde{f}_m\|$ $= \|\Psi(\pi(f_n) - \pi(f_m))\| = \|\pi(f_n) - \pi(f_m)\|_\infty \to 0$ for $n, m \to \infty$.
$\Rightarrow \pi(f_n)$ Cauchy in $L^\infty(E)$ $\Rightarrow \exists \pi(f) = \lim_{n\to\infty} \pi(f_n)$
$\Rightarrow \|\tilde{f} - \tilde{f}_n\| = \|\pi(f) - \pi(f_n)\| \xrightarrow{n\to\infty} 0 \Rightarrow T = \tilde{f} \Rightarrow T \in \mathcal{A}$.
$\therefore$ $\mathcal{A}$ is a closed normal subalgebra of $\mathcal{B}(\mathcal{H})$.

$(\varphi, \tilde{I}\psi) = (\varphi, \psi)$ $\forall \varphi, \psi \in \mathcal{H}$

Since $id_{\mathcal{H}} = \tilde{I} \Rightarrow id_{\mathcal{H}} \in \mathcal{A}$. Thus by 18.14. we have $\forall T \in \mathcal{A}$: $\sigma_{\mathcal{A}}(T) = \sigma_{\mathcal{B}(\mathcal{H})}(T) =: \sigma(T)$. Therefore, $\lambda \in \sigma(\tilde{f}) \Leftrightarrow \underbrace{\lambda \tilde{I} - \tilde{f}}_{= (\widetilde{\lambda 1 - f}) = \Psi(\pi(\lambda 1 - f))} \in G(\mathcal{A})$
$\Leftrightarrow \lambda \pi(1) - \pi(f) \in G(L^\infty(E)) \Leftrightarrow \lambda \in \sigma(\pi(f))$
and thus $\sigma(\tilde{f}) = \sigma(\pi(f)) = R_E(f)$ by 19.8.b). Proves "c)".

Suppose $S \in \mathcal{B}(\mathcal{H})$. If $S E(\omega) = E(\omega) S$ $\forall \omega \in M \Rightarrow S\tilde{s} = \tilde{s}S$ for any simple $s \Rightarrow S\tilde{f} = \lim_{n\to\infty} S\tilde{s}_n = \lim \tilde{s}_n S = \tilde{f}S$.
$\Rightarrow S$ commutes with $\mathcal{A}$. Since $\omega \in M \Rightarrow E(\omega) = \Psi(\pi(f))$ for $f(\lambda) = \mathbb{1}(\lambda \in \omega) \Rightarrow E(\omega) \in \mathcal{A}$ and the other direction is trivially true $\square$

**19.10. Definition:** (Symbolic calculus of normal operators)

Suppose $T \in \mathcal{B}(\mathcal{H})$ is normal, and let $E$ denote its spectral decomposition. Then for any bounded Borel function $f: \sigma(T) \to \mathbb{C}$ the operator $f(T) \in \mathcal{B}(\mathcal{H})$ is defined by $f(T) := \Psi(\pi(f))$, i.e., it is the unique operator for which

$$(\varphi, f(T)\psi) = \int_{\sigma(T)} E_{\varphi,\psi}(d\lambda)\, f(\lambda) \quad \forall \varphi, \psi \in \mathcal{H}.$$

* Note that Borel measurability is quite a weak condition: any $f(\lambda)$ which is a pointwise convergent limit of continuous functions is Borel measurable. [RCA, 1.14.a]

* In fact, boundedness is not completely necessary: for any Borel measurable $f$, $f(T)$ can be defined as a closed, densely defined, normal (but unbounded, unless $f \in L^\infty$) operator. [FA, 13.24, 13.25]

**19.11. Proposition:** Suppose $T \in \mathcal{B}(\mathcal{H})$ is normal and $f, g$ are bounded Borel functions on $\sigma(T)$. Then

a) $f(T)$ is normal and $f(T)^* = (f^*)(T)$.
b) $f(T)g(T) = g(T)f(T) = (fg)(T)$
c) $\alpha f(T) + \beta g(T) = (\alpha f + \beta g)(T)$ $\forall \alpha, \beta \in \mathbb{C}$
d) $\|f(T)\| = \|f\|_{E,\infty} \leq \|f\|_\infty$
e) $\sigma(f(T)) = R_E(f)$
f) If $S \in \mathcal{B}(\mathcal{H})$ and $ST = TS$, then $Sf(T) = f(T)S$.

In particular,

g) If $f(\lambda)=\alpha \ \forall \lambda \in \sigma(T)$, then $f(T) = \alpha 1$ $(\alpha \in \mathbb{C})$
h) If $f(\lambda)=\lambda \ \forall \lambda \in \sigma(T)$, then $f(T) = T$
i) If $z_0 \in \rho(T)$ and $f(\lambda) = (z_0-\lambda)^{-1}$, then
$f(T) = (z_0 1 - T)^{-1} = R_{z_0}(T)$

In addition,

j) If $f_n$, $n \in \mathbb{N}_+$, are bounded Borel functions and $\|f-f_n\|_\infty \to 0$, then $\|f(T)-f_n(T)\| \to 0$.
k) $\exists\, \tau_n$, $n \in \mathbb{N}_+$, each of which is a finite linear combination of mutually orthogonal projections, s.t. $\tau_n \to f(T)$.

<u>Proof:</u> Since $f(T) := \Phi(\pi(f))$, $g(T) := \Phi(\pi(g))$, "b)" and "c)" follow since $\Phi \circ \pi$ is a homomorphism. (15.9, & 17.4.)
By 15.9. a) $\Rightarrow f(T)^* = \Phi(\pi(f))^* = \Phi(\pi(f^*)) = (f^*)(T)$
$\Rightarrow f(T)^* f(T) \overset{b)}{=} f(T) f(T)^*$. Thus "a)" holds.
Applying 15.9, $\Rightarrow \|f(T)\| = \|\Phi(\pi(f))\| = \|\pi(f)\| \overset{15.8.}{=} \|f\|_{E,\infty}$ and $\|\pi(f)\| \le \|f\|_\infty$. Thus "d)" holds. "e)" follows from $\sigma(f(T)) = \sigma(\Phi(\pi(f))) = R_E(f)$. (by 15.9, c). If $S \in \mathcal{B}(\mathcal{H})$ and $ST = TS \overset{15.4.}{\Rightarrow} SE(\omega) = E(\omega)S \ \forall \omega \in M \overset{15.9.}{\Rightarrow} S\Phi(\pi(f)) = \Phi(\pi(f))S \Leftarrow$ "f)".
"g)" $\Rightarrow (\varphi, f(T)\psi) = \int E_{\varphi,\psi}(d\lambda)\, \alpha = \alpha\,(\varphi, E(\sigma(T))\psi) = \alpha(\varphi,\psi)$
$\Rightarrow f(T) = \alpha 1$.
"h)" follows from 15.4.
"i)" $z_0 \in \rho(T) \Rightarrow \exists\, \varepsilon > 0$ s.t. $|z_0 - \lambda| \ge \varepsilon \ \forall \lambda \in \sigma(T)$ [$\rho(T) = \sigma(T)^c =$ open]
$\Rightarrow f(\lambda) := (z_0-\lambda)^{-1}$ bounded & continuous $\Rightarrow$ Borel meas.
In addition, by the previous results $z_0 1 - T = g(T)$ for $g(\lambda) := z_0 - \lambda$. (c, g, h). Thus by "b)"
$f(T)\, g(T) = g(T)\, f(T) = (fg)(T)$ where
$(fg)(\lambda) = (z_0-\lambda)^{-1}(z_0-\lambda) = 1 \ \forall \lambda \in \sigma(T) \overset{g)}{\Rightarrow} f(T)g(T) = 1 = g(T)f(T)$
$\Rightarrow f(T) = g(T)^{-1} = (z_0 1 - T)^{-1}$.
"j)" $\|f(T) - f_n(T)\| = \|\Phi(\pi(f)) - \Phi(\pi(f_n))\| \overset{15.9.}{=} \|\pi(f) - \pi(f_n)\|$
$= \|\pi(f-f_n)\| \le \|f-f_n\|_\infty \to 0$.
"k)" As on p. 165, $\exists\, S_n$, $n \in \mathbb{N}_+$, simple functions s.t. $\|f - S_n\|_\infty \to 0$, $\Rightarrow \tilde{S}_n \to \tilde{f} = f(T)$. Each $\tilde{S}_n = \sum_{i=1}^{\sim} \alpha_i E(\omega_i)$ where $E(\omega_i)E(\omega_{i'}) = E(\omega_i \cap \omega_{i'}) = 0$ since $\omega_i$ and $\omega_{i'}$ are disjoint. $\square$