18. C^*-algebras

18.1. Definition: Suppose $\mathcal{A}$ is a complex algebra. A map $x \mapsto x^*$, $\mathcal{A} \to \mathcal{A}$, is called an involution if it satisfies $\forall x, y \in \mathcal{A}$, $\alpha \in \mathbb{C}$,

a) $(x+y)^* = x^* + y^*$ and $(\alpha x)^* = \alpha^* x^*$
(conjugate linear)
b) $(xy)^* = y^* x^*$ (antihomomorphic)
c) $x^{**} = x$ (has period 2)

An element $x \in \mathcal{A}$ satisfying $x^* = x$ is then called self-adjoint or hermitian. If $x^* x = x x^*$, then x is called normal.

18.2. Definition: $\mathcal{A}$ is called a C^*-algebra, if
a) it is a Banach algebra,
b) it has an involution $* : \mathcal{A} \to \mathcal{A}$,
and c) the involution satisfies $\|x^* x\| = \|x\|^2 \; \forall x \in \mathcal{A}$.

18.3. Proposition: Suppose $\mathcal{H}$ is a (complex) Hilbert space. Then the adjoint mapping $x \mapsto x^*$, is an involution on $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$ which makes $\mathcal{B}(\mathcal{H})$ into a C^*-algebra.

Proof: By Ex. 8.1.c) $\mathcal{B}(\mathcal{H})$ is a Banach space under the operator norm. Operator composition defines a multiplication on $\mathcal{B}(\mathcal{H})$ which makes it into a Banach algebra (for instance, $((x+y)z)(u) = (x+y)(z(u)) = x(z(u)) + y(z(u))$ $= (x \circ z)(u) + (y \circ z)(u) = (xz + yz)(u)$, and $\|(xy)(u)\|$ $= \|x(y(u))\| \leq \|x\| \, \|y(u)\| \leq \|x\| \, \|y\| \, \|u\| \;\; \forall u \in \mathcal{H}$ $\Rightarrow \|xy\| \leq \|x\| \, \|y\|$. Also $e = id_{\mathcal{H}}$ satisfies $\|e(u)\| = \|u\| \; \forall u \in \mathcal{H}$ $\Rightarrow \|e\| = 1$.) By Ex. 3.1, adjoint is an involution on $\mathcal{B}(\mathcal{H})$ and $\|x^* x\| = \|x\|^2$. $\square$

* There are also involutions which do not satisfy the C^*-property. They can nevertheless provide useful information, as the following result shows

18.4. Proposition: Suppose $\mathcal{A}$ is a Banach algebra with an involution. Consider some $x \in \mathcal{A}$.

a) The following elements are hermitian: $x + x^*$, $i(x - x^*)$, xx^*, and x^*x.
b) $\exists!\, u, v \in \mathcal{A}$ which are hermitian and $x = u + iv$.
c) the unit e is hermitian and normal.
d) x is invertible iff x^* is invertible, and then $(x^*)^{-1} = (x^{-1})^*$
e) $\lambda \in \sigma(x) \Leftrightarrow \lambda^* \in \sigma(x^*)$
f) If $\mathcal{A}$ is also commutative and semisimple, then the involution is a continuous map $\mathcal{A} \to \mathcal{A}$.

Proof. a) $(x \pm x^*)^* = x^* + (\pm x^*)^* = x^* \pm x$
$\Rightarrow (x + x^*)^* = x + x^*$ and $(i(x - x^*))^* = -i(x^* - x) = i(x - x^*)$.
Also $(xx^*)^* = (x^*)^* x^* = xx^*$ & $(x^*x)^* = x^*(x^*)^* = x^*x$ ■

b) Define $u := \frac{1}{2}(x + x^*)$, $v = -\frac{1}{2}i(x - x^*)$. By a) u, v are hermitian. Also $u + iv = \frac{1}{2}(x + x^* + (x - x^*))$
$= \frac{1}{2}(2x) = x$. If $x = u' + iv'$ with u', v' hermitian, set $w = v' - v \Rightarrow w^* = w$ and $iw = iv' - iv$
$= u' + iv' - u' - (u + iv) + u = u - u' + x - x = u - u'$
$\Rightarrow iw = (iw)^* = -iw^* = -iw \Rightarrow 2iw = 0 \Rightarrow w = 0$
$\Rightarrow iw = 0$. Thus $u' = u$ and $v' = v$. ■

c) As $e^* = e^*e \overset{a)}{\Rightarrow} e^* = (e^*)^* = e \Rightarrow e^*e \overset{c)}{=} ee = e = ee^*$ ■

d) $x \in G(\mathcal{A}) \Leftrightarrow \exists x^{-1} \in \mathcal{A}$ s.t. $xx^{-1} = e = x^{-1}x \Rightarrow e = e^* = x^*(x^{-1})^*$
$= (x^{-1})^* x^* \Rightarrow x^* \in G(\mathcal{A})$ and $(x^*)^{-1} = (x^{-1})^*$. Thus also
$x^* \in G(\mathcal{A}) \Rightarrow x = (x^*)^* \in G(\mathcal{A})$. ■

e) $\lambda \in \sigma(x) \Leftrightarrow \lambda e - x \notin G(\mathcal{A}) \overset{d)}{\Leftrightarrow} (\lambda e - x)^* \notin G(\mathcal{A}) \overset{c)}{\Leftrightarrow} \lambda^* e - x^* \notin G(\mathcal{A})$
$\Leftrightarrow \lambda^* \in \sigma(x^*)$ ■

f) Let Δ = Collection of complex homom. of $\mathcal{A}$, as before. If $h \in \Delta$, define $\phi(x) = (h(x^*))^* \overset{c)}{\Rightarrow} \phi(e) = (h(e))^* = 1$
$\Rightarrow \phi \neq 0$. Also $\phi(xy) = (h((xy)^*))^* = (h(y^*x^*))^*$
$= (h(y^*)h(x^*))^* = \phi(x)\phi(y)$, and $\phi(\alpha x + \beta y) = (h((\alpha x + \beta y)^*))^*$
$= (h(\alpha^* x^* + \beta^* y^*))^* = (\alpha^* h(x^*) + \beta^* h(y^*))^* = \alpha\phi(x) + \beta\phi(y)$.
Thus $\phi \in \Delta \overset{16.8.}{\Rightarrow} \phi$ is contin. If $x_n \to x$ and $x_n^* \to y$,

ϕ contin.

then $(h(x^*))^* = \phi(x) \overset{\phi\text{ contin.}}{=} \lim_{n\to\infty} \phi(x_n) = \lim_{n\to\infty} h(x_n^*)^*$

$\overset{*\text{ contin. on } \mathbb{C}}{=} (\lim_{n\to\infty} h(x_n^*))^* \overset{h\text{ contin.}}{=} h(y)^*$. Thus $h(y) = h(x^*)$ $\forall h \in \Delta$

$\Rightarrow y - x^* \in \operatorname{Ker} h$ $\forall h \in \Delta$ $\Rightarrow y - x^* \in \operatorname{rad} \mathcal{A} = \{0\} \Rightarrow y = x^*$.
Thus by the closed graph theorem (11.5 & 11.7.)
the real-linear map $x \mapsto x^*$ is continuous. $\square$

18.5. Proposition: Suppose $\mathcal{A}$ is a C^*-algebra. Then
$\|x^*\| = \|x\|$ and $\|x^* x\| = \|x^*\|\|x\| = \|xx^*\|$
for all $x \in \mathcal{A}$, and the map $x \mapsto x^*$ is continuous.

Proof: Since $\|x\|^2 = \|x^* x\| \leq \|x^*\|\|x\|$, we have $\|x\| \leq \|x^*\|$ for $x \neq 0$. Also $0 = \|0\| \leq \|0^*\|$. Thus $\|x\| \leq \|x^*\|$ $\forall x$
$\Rightarrow \|x^*\| \leq \|(x^*)^*\| = \|x\|$. $\therefore$ $\|x^*\| = \|x\|$ $\forall x \in \mathcal{A}$.
$\Rightarrow \|(x+h)^* - x^*\| = \|(x+h-x)^*\| = \|h^*\| = \|h\|$ $\forall x, h \in \mathcal{A}$
$\Rightarrow$ $*$ is continuous. Finally, $\|x^* x\| = \|x\|^2 = \|x^*\|\|x\|$
$= \|x^*\|^2 = \|(x^*)^* x^*\| = \|x x^*\|$ $\square$

18.6. Definition: Consider a complex algebra $\mathcal{A}$ with an involution, and suppose $S \subset \mathcal{A}$.

a) If $xy = yx$ $\forall x, y \in S$, then S <u>commutes</u>.
b) If S commutes and $x^* \in S$ $\forall x \in S$, then S is <u>normal</u>.
c) If S is normal, and no $S' \subset \mathcal{A}$, with $S \subsetneq S'$, is normal, S is called <u>maximally normal</u>.

18.7. Theorem: Suppose $\mathcal{A}$ is a Banach algebra with an involution, and $S \subset \mathcal{A}$ is normal.

a) S is contained is some maximally normal $M \subset \mathcal{A}$.
b) If S is maximally normal, it is a closed commutative subalgebra of $\mathcal{A}$. S is also then a commutative Banach algebra, and $\sigma_S(x) = \sigma_{\mathcal{A}}(x)$ $\forall x \in S$.

Proof: "a)" Follows from applying Zorn's lemma to the partial order "$\subset$" on $\mathcal{P} := \{S' \subset \mathcal{A} \mid S' \text{ normal and } S \subset S'\}$.
(The upper bound for a totally ordered $\mathcal{O}$ is given by $\bigcup_{S' \in \mathcal{O}} S'$.)

"b)" Assume that S is maximally normal. If $x \in A$ is normal, and $xy = yx \ \forall y \in S$, then necessarily $x \in S$: since S is normal $\Rightarrow xy^* = y^*x \ \forall y \in S \Rightarrow yx^* = x^*y \ \forall y \in S$ $\Rightarrow S' := S \cup \{x, x^*\}$ is normal $\overset{S \text{ maximal}}{\Longrightarrow} x \in S$.

Thus if $\alpha \in \mathbb{C}$, $x \in S \Rightarrow (\alpha x)^*(\alpha x) = |\alpha|^2 x^* x$ $= |\alpha|^2 x x^* = (\alpha x)(\alpha x)^* \Rightarrow \alpha x$ is normal & $\forall y \in S$: $(\alpha x) y = \alpha(xy) = \alpha(yx) = y(\alpha x)$. Thus $\alpha x \in S$. Similarly, $x, y \in S \Rightarrow (x+y)^*(x+y) = x^*x + x^*y + y^*x + y^*y$ $\overset{x^*, y^* \in S}{=} xx^* + yx^* + xy^* + yy^* = (x+y)(x+y)^*$ and $(xy)^* xy$ $= y^*x^*xy = y^*xx^*y = xy^*yx^* = xyy^*x^* = (xy)(xy)^*$ $\Rightarrow x+y$ and xy are normal and $\forall z \in S$: $(x+y)z$ $= xz + yz = zx + zy = z(x+y)$, $(xy)z = xzy = z(xy)$. $\Rightarrow x+y, xy \in S$. Thus S is a subalgebra of A, and $xy = yx \ \forall x, y \in S$. Suppose $x \in \overline{S} \Rightarrow \exists x_n \in S$, $n \in \mathbb{N}_+$, s.t. $x_n \to x \overset{16.2.}{\Rightarrow} \forall y \in S$: $xy = \lim_{n\to\infty}(x_n y) = \lim_{n\to\infty}(y x_n) = yx$ and thus also $x^*y = (y^*x)^* = (xy^*)^* = yx^* \ \forall y \in S$ $\Rightarrow x^*x = \lim_{n\to\infty}(x^* x_n) = \lim_{n\to\infty}(x_n x^*) = xx^*$. $\therefore x \in S$.

This proves that S is a commutative closed subalgebra of A. By 18.4. c), e is normal and as $ey = y = ye \ \forall y \in S$ $\Rightarrow e \in S$. Thus by 16.17. a), S is a commutative Banach algebra. Suppose $x \in G(A) \cap S \overset{(18.4.d)}{\Rightarrow} x^* \in G(A) \cap S$ and $(x^{-1})^* = (x^*)^{-1}$ $\Rightarrow x^{-1}(x^{-1})^* = x^{-1}(x^*)^{-1} = (x^*x)^{-1} \overset{x \text{ normal}}{=} (xx^*)^{-1} = (x^*)^{-1}x^{-1} = (x^{-1})^* x^{-1}$ $\Rightarrow x^{-1}$ is normal and $\forall y \in S$: $x^{-1}y = (x^{-1}y)(xx^{-1}) = x^{-1}yxx^{-1}$ $\overset{x, y \in S}{=} x^{-1}xyx^{-1} = yx^{-1}$. Thus $x^{-1} \in S \Rightarrow x \in G(S)$. As $G(S) \subset G(A)$ $\Rightarrow G(S) = G(A) \cap S$. Therefore, if $x \in S$, we have $\lambda \in \sigma_S(x) \Leftrightarrow \lambda e - x \notin G(S) \Leftrightarrow \lambda e - x \notin G(A)$ $\left(\text{as } x, e \in S, \lambda \in \mathbb{C} \Rightarrow \lambda e - x \in S\right)$ $\Leftrightarrow \lambda \in \sigma_A(x)$. $\therefore \sigma_S(x) = \sigma_A(x)$ $\square$

18.8. Definition: Suppose A is a Banach algebra and $E \subset A$. The closure of the collection of all polynomials formed by elements of E is called the Banach subalgebra generated by E.

* For explicit definition, see Ex. 11.2. which also shows that the closure forms a Banach algebra.

* The following theorem starts to bind the previous results together:

18.9. Proposition: Suppose $\mathcal{A}$ is a C^*-algebra.

If $E \subset \mathcal{A}$ is <u>normal</u>, then the Banach subalgebra $\mathcal{A}_E \subset \mathcal{A}$ generated by E is a commutative C^*-algebra, and a normal subalgebra of $\mathcal{A}$.

<u>Proof.</u> By Ex. 11.2., $\mathcal{A}_E$ is a commutative Banach subalgebra and $e \in \mathcal{A}_E$. If $x \in \mathcal{A}_E \Rightarrow \exists y_n \in \operatorname{span}(F)$ s.t. $y_n \to x$ where $F = \{e\} \cup \{x_1 \cdots x_N \mid x_i \in E\}$. $\Rightarrow$ $y_n = \sum_{l=1}^{M_n} \alpha_{l,n} \prod_{i=1}^{N_{l,n}} \underbrace{x_{i,l,n}}_{\in E}$

$\Rightarrow y_n^* = \sum_{l=1}^{M_n} \alpha_{l,n}^* \prod_{i=N_{l,n}}^{1} x_{i,l,n}^*$ where $N_{l,n} = 0 \Rightarrow \prod_{i=1}^{N_{l,n}} x_{i,l,n} := e \overset{18.4.c)}{\Rightarrow} e^* = e \in F$ or $N_{l,n} > 0$ and $x_{i,l,n} \in E \Rightarrow x_{i,l,n}^* \in E \Rightarrow \prod_{i=N_{l,n}}^{1} x_{i,l,n}^* \in F$. Thus always $y_n^* \in \operatorname{span}(F)$. Since $\|x^* - y_n^*\| = \|(x-y_n)^*\| \overset{18.5.}{=} \|x - y_n\| \xrightarrow{n\to\infty} 0$, we thus have $x^* \in \overline{\operatorname{span}(F)} = \mathcal{A}_E$. $\therefore$ $\mathcal{A}_E$ is normal, and $*: \mathcal{A}_E \to \mathcal{A}_E$ is an involution for which $\|x^* x\| = \|x\|^2$ $\forall x \in \mathcal{A}_E$. $\therefore$ $\mathcal{A}_E$ is a commutative C^*-algebra $\square$

18.10. $\to$

18.11. Theorem: Suppose $\mathcal{H}$ is a (complex) Hilbert space and consider the C^*-algebra $\mathcal{B}(\mathcal{H})$. Assume $E_0 \subset \mathcal{B}(\mathcal{H})$ <u>commutes</u> and every $x \in E_0$ is <u>normal</u>. Denote $E := E_0 \cup E_0^*$, where $E_0^* := \{x^* \mid x \in E_0\}$, and let $\mathcal{A}_0$ denote the Banach subalgebra generated by E. Then

a) E is normal

b) $\mathcal{A}_0$ is a commutative C^*-subalgebra of $\mathcal{B}(\mathcal{H})$

c) If E_0 is <u>finite</u> and <u>nonempty</u>, then the maximal ideal space Δ_0 of $\mathcal{A}_0$ is homeomorphic to a compact subset of $\mathbb{C}^{E_0}$ which is contained in $\prod_{x \in E_0} \sigma(x)$; the map $\Phi: h \mapsto (h(x))_{x \in E_0}$ provides a homeomorphism. In particular, if $E_0 = \{x_0\}$, $x_0 \in \mathcal{B}(\mathcal{H})$, then Δ_0 is homeomorphic to $\sigma(x_0)$.

Denote $K := \Phi(\Delta_0) \subset \mathbb{C}^{E_0}$, and define $\Psi: C(K) \to \mathcal{A}_0$ by $\Psi(f)^\wedge := f \circ \Phi \in C(\Delta_0)$ $\forall f \in C(K)$. Then Ψ is an isometric isomorphism <u>onto</u> $\mathcal{A}_0$ and $\Psi(f^*) = (\Psi(f))^*$ $\forall f \in C(K)$. Moreover, if $f_0(z) = z_{x_0}$ $\forall z \in K$ and some $x_0 \in E_0$ $\Rightarrow \Psi(f_0) = x_0$.

Proof: "a)" Follows from theorem 18.10." ; If $x, y \in E_0$
$\Rightarrow x, y$ are normal and $xy = yx \overset{18.10''}{\Rightarrow} x^* y = y x^*$ and $y^* x = x y^*$ and $x^* y^* = (yx)^* = (xy)^* = y^* x^*$. Thus $E = E_0 \cup E_0^*$ commutes and $x \in E \Rightarrow x^* \in E$. ∎
"b)" Follows from a) and 18.9. ∎
"c)" Let Δ_0 denote the maximal ideal space of A_0 which is nonempty, and define $\Phi: \Delta_0 \to \mathbb{C}^{E_0}$ by $\Phi(h)_x := h(x)$ $\forall x \in E_0 \Rightarrow \Phi(h)_x = \hat{x}(h)$ $\forall x \in E_0$. As each component is a continuous map $\Delta_0 \to \mathbb{C}$ (by def. of Gelfand topology) $\overset{|E_0| < \infty}{\Rightarrow}$ Φ is a continuous map onto $K := \Phi(\Delta_0)$. Moreover, by 17.8.c) $\Phi(h)_x \in \sigma(x)$ $\forall x \in E_0 \Rightarrow K \subset \prod_{x \in E_0} \sigma(x)$, and by 17.8.a) Δ_0 is compact $\Rightarrow K$ is compact. Suppose then that $h_1, h_2 \in \Delta_0$ and $\Phi(h_1) = \Phi(h_2) \Rightarrow h_1(x) = h_2(x)$ $\forall x \in E_0 \overset{18.10.}{\Rightarrow}$ $h_1(x^*) = h_1(x)^* = h_2(x)^* = h_2(x^*)$. Thus then $h_1(x) = h_2(x)$ $\forall x \in E$. Since h_1, h_2 are complex homom. $\Rightarrow h_1(e) = 1 = h_2(e)$ and (by induction) $h_1(x_1 \cdots x_n) = \prod_{i=1}^n h_1(x_i) = \prod_{i=1}^n h_2(x_i)$ $= h_2(x_1 \cdots x_n)$ $\forall x_i \in E, i = 1, \dots, n$. Since both h_1 and h_2 are linear, this shows that $h_1 = h_2$ on $\operatorname{Span}(F)$, which is dense in A_0. As h_1 and h_2 are contin. (16.8.) $\Rightarrow h_1 = h_2$ on A_0. Thus Φ is a continuous bijection between compact Hausdorff spaces Δ_0 and K $\Rightarrow$ Φ is a homeomorphism. (Suppose $C \subset \Delta_0$ is closed $\overset{\Delta_0 \text{ compact}}{\Rightarrow} C$ compact $\overset{\Phi \text{ contin.}}{\Rightarrow} \Phi(C)$ compact in $K \overset{K \text{ Hausd.}}{\Rightarrow} \Phi(C)$ closed in K. As $(\Phi^{-1})^{\leftarrow}(C) \overset{\Phi \text{ bij.}}{=} \Phi(C)$ this shows that Φ^{-1} is contin.)

If $E_0 = \{x_0\}$, 17.8.c) $\Rightarrow \Phi(\Delta_0) = \hat{x}_0(\Delta_0) = \sigma(x_0)$.

By 18.10. the Gelfand transform $G: A_0 \to C(\Delta_0)$, $G(x) = \hat{x}$, an isometric $*$-isomorphism. If $f \in C(K) \Rightarrow f \circ \Phi \in C(\Delta_0)$ $\Rightarrow \Psi(f) := G^{-1}(f \circ \Phi) \in A_0$. Also $\|\Psi(f)\| = \|f \circ \Phi\|_\infty$ $= \max_{h \in \Delta_0} |f(\Phi(h))| = \max_{z \in K = \Phi(\Delta_0)} |f(k)| = \|f\|_\infty \Rightarrow \Psi$ is an isometry.

If $x \in A_0 \Rightarrow G(x) \in C(\Delta_0) \Rightarrow f := G(x) \circ \Phi^{-1} \in C(K)$ and $f \circ \Phi = G(x)$ $\Rightarrow \Psi(f) = x$. Thus Ψ is onto. If $\alpha, \beta \in \mathbb{C}$, $f_1, f_2 \in C(K)$ $\Rightarrow (\alpha f_1 + \beta f_2) \circ \Phi = \alpha f_1 \circ \Phi + \beta f_2 \circ \Phi \overset{G^{-1} \text{ lin.}}{\Rightarrow} \Psi(\alpha f_1 + \beta f_2) = \alpha \Psi(f_1) + \beta \Psi(f_2)$
Similarly, $((f_1 f_2) \circ \Phi)(h) = f_1(\Phi(h)) f_2(\Phi(h)) = ((f_1 \circ \Phi)(f_2 \circ \Phi))(h)$ $\forall h \in \Delta_0$ $\overset{G^{-1} \text{ homom}}{\Rightarrow}$ $\Psi(f_1 f_2) = \Psi(f_1) \Psi(f_2)$. Finally, if $\Psi(f_1) = \Psi(f_2)$ $\Rightarrow f_1 \circ \Phi = f_2 \circ \Phi \Rightarrow f_1 = f_2$. Thus Ψ is a bijection, i.e. Ψ = isometric isomorphism. $x = \Psi(f) \Rightarrow G(x^*) = G(x)^* = (f \circ \Phi)^*$ $\Rightarrow G(x^*)(h) = f(\Phi(h))^* = f^*(\Phi(h))$ $\forall h \Rightarrow G(x^*) = f^* \circ \Phi \Rightarrow x^* = \Psi(f^*)$.
If $x_0 \in E_0 \Rightarrow G(x_0)(h) = h(x_0) = \Phi(h)_{x_0} = (f_0 \circ \Phi)(h) \Rightarrow x_0 = \Psi(f_0)$ as $f_0 \in C(K)$ □

18.10. Theorem (Gelfand–Naimark)

Suppose $\mathcal{A}$ is a commutative C^*-algebra, with maximal ideal space Δ. The Gelfand transform $\mathcal{G}: x \mapsto \hat{x}$ is then an isometric $*$-isomorphism of $\mathcal{A}$ onto $C(\Delta)$. (This means that $\mathcal{G}$ is an isomorphism, isometry, and $\mathcal{G}(x^*) = \mathcal{G}(x)^*$ $\forall x \in \mathcal{A}$.) In addition, $x \in \mathcal{A}$ is self-adjoint iff. $\mathcal{G}(x) = \hat{x}$ is real-valued.

Proof: Assume first that $x \in \mathcal{A}$, $x^* = x$, and $h \in \Delta$. Set

$y_t := x + ite$, $t \in \mathbb{R}$, and $\alpha = \operatorname{Re} h(x)$, $\beta = \operatorname{Im} h(x)$.

$\Rightarrow h(x) = \alpha + i\beta \Rightarrow h(y_t) = \alpha + i\beta + it\,h(e) = \alpha + i(\beta + t)$

$\Rightarrow |h(y_t)|^2 = \alpha^2 + (\beta+t)^2$ $\forall t \in \mathbb{R}$. But $y_t^* y_t = (x^* - ite)(x + ite)$

$= x^*x + itx^* - itx + t^2 e \overset{x^*=x}{=} x^*x + t^2 e$ and thus by 16.8.

$\Rightarrow \alpha^2 + (\beta+t)^2 = |h(y_t)|^2 \le \|y_t\|^2 \overset{C^*}{=} \|y_t^* y_t\| \le \|x^*x\| + t^2 = \|x\|^2 + t^2$

$\Rightarrow \alpha^2 + \beta^2 + 2\beta t \le \|x\|^2$ $\forall t \in \mathbb{R}$ $\Rightarrow \beta = 0 \Rightarrow \hat{x}(h) = h(x) \in \mathbb{R}$ $\forall h$

$\Rightarrow \mathcal{G}(x)$ is real-valued. (proves "$\Rightarrow$" in the final statement)

If $x \in \mathcal{A} \overset{18.4.b)}{\Rightarrow} x = u + iv$, u, v self-adjoint, $\overset{\mathcal{G}\text{ homom.}}{\Rightarrow} \mathcal{G}(x) = \mathcal{G}(u) + i\mathcal{G}(v)$

$\Rightarrow \mathcal{G}(x)(h) = \underbrace{\hat{u}(h)}_{\in \mathbb{R}} + i\underbrace{\hat{v}(h)}_{\in \mathbb{R}} \Rightarrow \mathcal{G}(x)(h)^* = \hat{u}(h) - i\hat{v}(h) = \mathcal{G}(u - iv)(h)$

$\Rightarrow \mathcal{G}(x)^* = \mathcal{G}(u - iv) = \mathcal{G}(x^*)$. Thus $\mathcal{G}$ is a **$*$-homomorphism** onto $\hat{\mathcal{A}}$ which is a subalgebra of $C(\Delta)$. For any $\alpha \in \mathbb{C}$, $\alpha e \in \mathcal{A}$ and $\mathcal{G}(\alpha e)(h) = h(\alpha e) = \alpha h(e) = \alpha$ $\forall h \in \Delta$ $\Rightarrow \hat{\mathcal{A}}$ contains constant functions. In addition, $h_1 \ne h_2 \Rightarrow \exists x \in \mathcal{A}$ s.t. $h_1(x) \ne h_2(x)$ and thus $\hat{\mathcal{A}}$ separates points on Δ. Thus $\overline{\hat{\mathcal{A}}}$ is a closed subalgebra of $C(\Delta)$, which contains contains constants and separates points on Δ. (16.17.a) If $f \in \overline{\hat{\mathcal{A}}}$, $\exists x_n \in \mathcal{A}$ s.t. $\hat{x}_n \to f$ $\Rightarrow \hat{x}_n^* \to f^*$, and since $\hat{x}_n^* = \mathcal{G}(x_n)^* = \mathcal{G}(x_n^*) \in \hat{\mathcal{C}} \Rightarrow f^* \in \overline{\hat{\mathcal{A}}}$.

Therefore, $\overline{\hat{\mathcal{A}}}$ satisfies the assumptions of the Stone-Weierstrass theorem $\Rightarrow \overline{\hat{\mathcal{A}}} = C(\Delta)$ i.e. $\hat{\mathcal{A}}$ is dense in $C(\Delta)$.

Suppose then that $x \in \mathcal{A}$ and set $y := x^*x \overset{18.4.a)}{\Rightarrow} y^* = y$

$\Rightarrow \|y^2\| = \|y^* y\| \overset{C^*}{=} \|y\|^2 \Rightarrow \|y^m\| = \|y\|^m$ $\forall m = 2^n$, $n \in \mathbb{N}_+$

(Proof: induction, $\|y^{2m}\| = \|y^m y^m\| = \|(y^m)^* y^m\| = \|y^m\|^2 = \|y\|^{2m}$.)

$\overset{16.14.c)}{\Rightarrow} r_\sigma(y) = \lim_{n\to\infty} \|y^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|y^{2^n}\|^{2^{-n}} = \|y\| \overset{17.8.c)}{\Rightarrow} \|\hat{y}\|_\infty = r_\sigma(y) = \|y\|$.

But $\hat{y}(h) = h(y) = h(x^*x) = h(x^*)h(x) = |\hat{x}(h)|^2$ since now $h(x^*) = \mathcal{G}(x^*)(h) = \mathcal{G}(x)^*(h) = h(x)^*$. Thus $\|\hat{x}\|_\infty^2 = \sup_{h \in \Delta} |\hat{x}(h)|^2$ $= \|\hat{y}\|_\infty = \|y\| = \|x^*x\| = \|x\|^2 \Rightarrow \|\mathcal{G}(x)\|_\infty = \|x\|$.

Thus $\mathcal{G}$ is an isometry $\Rightarrow \|\mathcal{G}(x) - \mathcal{G}(x')\|_\infty = \|x - x'\| \Rightarrow \mathcal{G}$ is 1-1

(*) Thus if $\mathcal{G}(x)(h) \in \mathbb{R}\ \forall h \Rightarrow \mathcal{G}(x) = \mathcal{G}(x)^* = \mathcal{G}(x^*) \Rightarrow x = x^*$. Proves "$\Leftarrow$" in the final statement.

Thus $\mathcal{G}$ is bijective homomorphism $\Rightarrow$ $\mathcal{G}$ is isomorphism.
$\therefore$ $\mathcal{G}$ is isometric isomorphism, and $\mathcal{G}(x^*) = \mathcal{G}(x)^* \Rightarrow$ $*$-isomorphism.(*) Finally, if $x_n \in \mathcal{A}$ is such that $\mathcal{G}(x_n) \to f$ in $C(\Delta)$, then $\|x_m - x_n\| = \|\mathcal{G}(x_m - x_n)\|_\infty = \|\mathcal{G}(x_m) - \mathcal{G}(x_n)\|_\infty$ $\Rightarrow$ (x_n) in Cauchy in $\mathcal{A}$ $\Rightarrow \exists x$ s.t. $x_n \to x$ and $\|\mathcal{G}(x) - \mathcal{G}(x_n)\| = \|x - x_n\| \to 0 \Rightarrow \mathcal{G}(x_n) \to \mathcal{G}(x)$ and thus $f = \mathcal{G}(x) \in \hat{\mathcal{A}}$.
Therefore, $\hat{\mathcal{A}}$ is closed $\Rightarrow \hat{\mathcal{A}} = C(\Delta)$. $\square$

18.10.' Theorem (Stone-Weierstrass)

Suppose K is a non-empty compact Hausdorff space, and consider the Banach algebra $C(K)$. If $\mathcal{A} \subset C(K)$ satisfies

a) $\mathcal{A}$ is a subalgebra
b) $\mathcal{A}$ is closed in $C(K)$
c) The constant map $a \mapsto 1$ belongs to $\mathcal{A}$
d) $\mathcal{A}$ separates points on K,
and e) $f \in \mathcal{A} \Rightarrow f^* \in \mathcal{A}$,

then $\mathcal{A} = C(K)$.

We will return to its proof later. (We will prove the more general Bishop's theorem.) As will we prove

18.10." Theorem (Fuglede-Putnam-Rosenblum)

Suppose $\mathcal{H}$ is a (complex) Hilbert space, and $M, N, T \in \mathcal{B}(\mathcal{H})$. If M, N are <u>normal</u>, and $MT = TN$, then $M^*T = TN^*$.