16.21. Theorem: Suppose $\mathcal{A}$ is a Banach algebra and $x \in \mathcal{A}$. If $\Omega \subset \mathbb{C}$ is open, and $\sigma(x) \subset \Omega$, then $\exists \delta > 0$ such that $\sigma(x+y) \subset \Omega$ for all $y \in \mathcal{A}$ with $\|y\| < \delta$.

Proof: By 16.14., the set $\rho(x) = \sigma(x)^c$ is open and nonempty, and the map $\lambda \mapsto (\lambda e - x)^{-1}$ is strongly holom. on $\rho(x)$ $\Rightarrow$ continuous on $\rho(x)$. Thus $\lambda \mapsto \|(\lambda e - x)^{-1}\|$ is a contin. map $\rho(x) \to [0, \infty)$. If $|\lambda| \geq 1 + 2\|x\| \Rightarrow \|\lambda^{-1} x\| \leq \frac{1}{1+2\|x\|} \|x\| \leq \frac{1}{2}$ and by 16.7. a), $\|(e - \lambda^{-1}x)^{-1}\| \leq \frac{1}{1 - \frac{1}{2}} = 2$. Thus

$$\|(\lambda e - x)^{-1}\| = \|\lambda^{-1}(e - \lambda^{-1}x)^{-1}\| = \frac{1}{|\lambda|} \|(e - \lambda^{-1}x)^{-1}\| \leq 2 \quad \forall |\lambda| > 1 + 2\|x\|.$$

On the other hand, $\overline{B(0, 1+2\|x\|)} \cap \Omega^c$ is compact and contained in $\rho(x)$. ($\sigma(x) \subset \Omega \Rightarrow \sigma(x)^c = \rho(x) \supset \Omega^c$). Thus $\exists M > 0$ s.t. $\|(\lambda e - x)^{-1}\| \leq M$ $\forall \lambda \notin \Omega$. Set $\delta := \frac{1}{M} > 0$, and assume $y \in \mathcal{A}$ with $\|y\| < \delta$. If $\lambda \in \Omega^c$, then $\lambda e - (x+y)$ $= (\lambda e - x)(e - (\lambda e - x)^{-1} y)$ where $\lambda e - x \in G(\mathcal{A})$ as $\lambda \in \rho(x)$ and $e - (\lambda e - x)^{-1} y \in G(\mathcal{A})$ as $\|(\lambda e - x)^{-1} y\| \leq M \cdot \|y\| < 1$. (Apply 16.7.a) $\Rightarrow \lambda e - (x+y) \in G(\mathcal{A}) \Rightarrow \lambda \in \sigma(x+y)^c$. $\therefore \sigma(x+y) \subset \Omega$ $\square$

17. Gelfand theory of commutative Banach algebras

17.1. Definition: Suppose $\mathcal{C}$ is a commutative complex algebra. $J \subset \mathcal{C}$ is called an ideal if

a) J is a subspace, and
b) $xy \in J$ $\forall x \in \mathcal{C}, y \in J$

* If J is an ideal, and $J \neq \mathcal{C}$, it is a proper ideal.
* If J is a proper ideal which is not contained in any larger proper ideal, it is called a maximal ideal. (Then, if J' is an ideal and $J \subset J'$, we must have $J' = J$ or $J' = \mathcal{C}$.)

17.2. Proposition:

a) If J is a proper ideal in a commutative complex algebra $\mathcal{C}$ and $x \in \mathcal{C}$ is invertible, then $x \notin J$.
b) If J is an ideal in a commutative Banach algebra, then $\bar{J}$ is also an ideal.

Proof: Exercise 11.1. □

17.3. Theorem

a) If $\mathcal{C}$ is a commutative complex algebra with unit, then every proper ideal of $\mathcal{C}$ is contained in a maximal ideal of $\mathcal{C}$.
b) If $\mathcal{C}$ is a commutative Banach algebra, then every maximal ideal of $\mathcal{C}$ is closed.

Proof: "a)" Let $J_0 \subset \mathcal{C}$ be a proper ideal, and $\mathcal{P}$ denote the collection of proper ideals J for which $J_0 \subset J$. "$\subset$" defines a partial order on $\mathcal{P}$. (See p. 107) Suppose $\mathcal{O}$ is totally ordered nonempty subset of $\mathcal{P}$, and define $J' := \bigcup_{J \in \mathcal{O}} J$. Now the unit $e \in \mathcal{C}$ is invertible $\overset{17.2.a)}{\Longrightarrow}$ $e \notin J \;\forall J \in \mathcal{O} \Rightarrow e \notin J'$. Thus $J' \neq \mathcal{C}$, and obviously $J_0 \subset J'$. If $\alpha, \beta \in \mathbb{C}$, $x, y \in J' \Rightarrow \exists J_1, J_2 \in \mathcal{O}$ s.t. $x \in J_1$, $y \in J_2$ and either $J_1 \subset J_2$ or $J_2 \subset J_1$.
$\Rightarrow \exists J \in \mathcal{O}$ s.t. $x, y \in J \Rightarrow \alpha x + \beta y \in J \subset J'$. Thus J' is a subspace. If also $z \in \mathcal{C}$, then $zy \in J_2$ (= ideal) $\Rightarrow zy \in J'$. Thus J' is a proper ideal with $J_0 \subset J'$ $\Rightarrow J' \in \mathcal{P}$ is an upper bound for $\mathcal{O}$. Thus by Zorn's lemma $\Rightarrow \exists$ maximal element $J \in \mathcal{P}$. ■

"b)" Suppose $J \subset \mathcal{C}$ is a maximal ideal. By 17.2.a) then $J \subset G(\mathcal{C})^c$ = closed $\Rightarrow \bar{J} \subset G(\mathcal{C})^c \Rightarrow e \notin \bar{J} \Rightarrow \bar{J} \neq \mathcal{C}$. By 17.2.b), $\bar{J}$ is an ideal, and thus it is a proper ideal. Since $J \subset \bar{J}$, this implies $\bar{J} = J$. □

17.4. Theorem: Suppose $\mathcal{C}$ and $\mathcal{C}_2$ are commutative Banach algebras.

a) If $\Phi: \mathcal{C} \to \mathcal{C}_2$ is a homomorphism, then $\ker \Phi$ is an ideal in $\mathcal{C}$, which is closed in case Φ is continuous

b) If J is a proper closed ideal in $\mathcal{C}$, then the quotient space $\mathcal{C}/J$ is also a commutative Banach algebra and the quotient map $\pi: \mathcal{C} \to \mathcal{C}/J$ is a continuous homomorphism, with $\ker \pi = J$.

Proof: "a)" Denote $J := \ker \Phi$. Since Φ is linear, J is a subspace (5.1.). If $x \in \mathcal{C}$, $y \in J \Rightarrow \Phi(xy) = \Phi(x)\Phi(y) = \Phi(x) 0_{\mathcal{C}_2} = 0_{\mathcal{C}_2} \Rightarrow xy \in \ker \Phi = J$. Thus J is an ideal.

"b)" Since J is a closed subspace $\Rightarrow \mathcal{C}_2 := \mathcal{C}/J$ is a Banach space (Theorem 9.3. f) and π is linear, continuous and $\ker \pi = J$. (9.2. & 9.3.) Analogously to the linear structure we define $\pi(x)\pi(y) := \pi(xy)$ $\forall x, y \in \mathcal{C}$; this yields a map $\mathcal{C}_2 \times \mathcal{C}_2 \to \mathcal{C}_2$ since, if $x' \sim x$ and $y' \sim y$, then $x' - x \in J$, $y' - y \in J \Rightarrow x'y' - xy = x'(y'-y) + (x'-x)y = x'(y'-y) + y(x'-x) \in J + J \subset J \Rightarrow x'y' \sim xy$ and thus $\pi(x'y') = \pi(xy)$. In addition, π is now obviously a homomorphism.

$\Rightarrow$ The unit of $\mathcal{C}_2$ is $\tilde{e} := \pi(e)$ and $\mathcal{C}_2$ is a commutative algebra: [Proof: ① $\pi(x)\pi(e) = \pi(xe) = \pi(x) = \pi(ex) = \pi(e)\pi(x)$.
② $\pi(x)\pi(y) = \pi(xy) = \pi(yx) = \pi(y)\pi(x)$. ③ $\pi(x)(\pi(y)\pi(z)) = \pi(x)\pi(yz) = \pi(x(yz)) = \pi((xy)z) = \pi(xy)\pi(z) = (\pi(x)\pi(y))\pi(z)$
④ $(\pi(x) + \pi(y))\pi(z) = \pi(x+y)\pi(z) = \pi((x+y)z) = \pi(xz + yz) = \pi(xz) + \pi(yz) = \pi(x)\pi(z) + \pi(y)\pi(z)$ $\Rightarrow \pi(x)(\pi(y) + \pi(z)) = (\pi(y) + \pi(z))\pi(x) = \pi(y)\pi(x) + \pi(z)\pi(x) = \pi(x)\pi(y) + \pi(x)\pi(z)$.
⑤ $\alpha(\pi(x)\pi(y)) = \alpha\pi(xy) = \pi(\alpha(xy)) = \pi((\alpha x)y) = \pi(\alpha x)\pi(y) = (\alpha\pi(x))\pi(y) \Rightarrow \pi(x)(\alpha\pi(y)) = (\alpha\pi(y))\pi(x) = \alpha(\pi(y)\pi(x)) = \alpha(\pi(x)\pi(y))$ $_\square$]

The norm of $\mathcal{C}_2$ is defined by $\|\pi(x)\| = \inf\{\|x-y\| \mid y \in J\} \leq \|x\|$. (9.3. e). Thus for all $x, x' \in \mathcal{C}$, $\varepsilon > 0$, $\exists y, y' \in J$ s.t. $\|\pi(x)\| + \varepsilon > \|x-y\|$ and $\|\pi(x')\| + \varepsilon > \|x'-y'\|$.

Since $(x-y)(x'-y') = xx' - xy' - yx' + yy' = xx' + \underbrace{(-x)y'}_{\in J} + \underbrace{(-x')y}_{\in J} + \underbrace{yy'}_{\in J}$
$\Rightarrow (x-y)(x'-y') \in xx' - J$, and thus

$\Rightarrow \|\pi(xx')\| \leq \|(x-y)(x'-y')\| \leq \|x-y\|\,\|x'-y'\| \leq (\|\pi(x)\|+\varepsilon)(\|\pi(x')\|+\varepsilon)\ \forall \varepsilon>0$
$\Rightarrow \|\pi(xx')\| \leq \|\pi(x)\|\,\|\pi(x')\|\ \forall x, x' \in \mathcal{C}$. Finally, as J is a proper ideal, $e \notin J \Rightarrow e \not\sim 0 \Rightarrow \tilde{e} \neq 0_{\mathcal{C}_2} \Rightarrow \|\tilde{e}\| > 0$. On the other hand, $\|\tilde{e}\| = \|\tilde{e}\tilde{e}\| \leq \|\tilde{e}\|\,\|\tilde{e}\| \Rightarrow 1 \leq \|\tilde{e}\| = \|\pi(e)\| \leq \|e\| = 1 \Rightarrow \|\tilde{e}\| = 1$. $\therefore \mathcal{C}_2$ is a Banach algebra $\square$

17.5. Theorem: Suppose $\mathcal{C}$ is a commutative Banach algebra, and let Δ denote the collection of all complex homomorphisms of $\mathcal{C}$.

a) If M is a maximal ideal of $\mathcal{C}$, then $M = \ker h$ for some $h \in \Delta$.
b) If $h \in \Delta$, then $\ker h$ is a maximal ideal of $\mathcal{C}$.
c) $x \in \mathcal{C}$ is invertible $\Leftrightarrow h(x) \neq 0\ \forall h \in \Delta$
d) $x \in \mathcal{C}$ is invertible $\Leftrightarrow$ x lies in no proper ideal of $\mathcal{C}$.
e) $\lambda \in \sigma(x) \Leftrightarrow h(x) = \lambda$ for some $h \in \Delta$.

Proof: "a)" Let M be a maximal ideal of $\mathcal{C}$. $\overset{17.3.}{\Rightarrow}$ M is closed $\overset{17.4.b)}{\Rightarrow}$ $\mathcal{C}/M$ is a Banach algebra. If $x_0 \in \mathcal{C}$, but $x_0 \notin M$, define $J := \{ax_0 + y \mid a \in \mathcal{C}, y \in M\}$. Then J is an ideal in $\mathcal{C}$: $\alpha(ax_0+y) + \beta(a'x_0+y') = (\alpha a + \beta a')x_0 + \alpha y + \beta y' \in J$ and $a'(ax_0+y) = (a'a)x_0 + a'y \in J\ \forall a, a' \in \mathcal{C}, y \in M (=\text{ideal})$. If $y \in M \Rightarrow y = 0x_0 + y \in J$, but also $x_0 = ex_0 + 0 \in J$ $\Rightarrow M \subsetneq J$. Since M is maximal, then $J = \mathcal{C}$. $\Rightarrow e \in J$ $\Rightarrow \exists a_0 \in \mathcal{C}, y_0 \in M$ s.t. $a_0x_0 + y_0 = e$. If $\pi: \mathcal{C} \to \mathcal{C}/M$ denotes the quotient map, then $\pi(e) = \pi(a_0x_0) = \pi(a_0)\pi(x_0) = \pi(x_0)\pi(a_0)$ (17.4.). $\Rightarrow \pi(x_0)$ is invertible. If $x_0 \in M$, then $\pi(x_0) = \pi(0)$. Thus every nonzero element in $\mathcal{C}/M$ is invertible $\overset{16.16.}{\Rightarrow} \exists$ isomorphism $\Phi: \mathcal{C}/M \to \mathbb{C}$. Set $h := \Phi \circ \pi$, when by 17.4.b) h is a homomorphism. $(h(xy) = \Phi(\pi(xy)) = \Phi(\pi(x)\pi(y)) = \Phi(\pi(x))\,\Phi(\pi(y)) = h(x)h(y))$ In addition, $h(x) = 0 \Leftrightarrow \Phi(\pi(x)) = 0 \Leftrightarrow \pi(x) = \Phi^{-1}(0) = \pi(0)$ $\Leftrightarrow x \sim 0 \Leftrightarrow x \in M$. Since $M \neq \mathcal{C}$, it follows that $h \not\equiv 0$. $\therefore h \in \Delta$ and $\ker h = M$ $\blacksquare$

"b)" $h \in \Delta \overset{17.4.a)}{\Rightarrow} J := \ker h$ is a closed ideal. By 16.6. $h(e) = 1 \Rightarrow e \notin J \Rightarrow J$ is proper. However, if $x \notin J$, then $h(x - h(x)e) = h(x) - h(x)h(e) = h(x) - h(x) = 0 \Rightarrow x - h(x)e \in J$.

If M is an ideal with $J \subsetneq M \Rightarrow \exists\, x_0 \in M$ s.t. $h(x_0) \neq 0$

$\Rightarrow e = \frac{1}{h(x_0)}(h(x_0)e - x_0 + x_0) = -\frac{1}{h(x_0)}\underbrace{(x_0 - h(x_0)e)}_{\in J \subset M} + \frac{1}{h(x_0)} x_0 \in M$

$\overset{17.2.a)}{\Rightarrow} M = \mathcal{C}$. Thus J is a maximal ideal. ■

"c)" If $x_0 \in \mathcal{C}$ is invertible and $h \in \Delta \overset{16.6.}{\Rightarrow} h(x_0) \neq 0$.

If $x_0 \in \mathcal{C}$ is not invertible, define $J := \{a x_0 \mid a \in \mathcal{C}\}$.

$\Rightarrow e \notin J$ (else $\exists a$ s.t. $e = a x_0 = x_0 a$ ↯), and J is an ideal ($\alpha(a x_0) + \beta(a' x_0) = (\alpha a + \beta a') x_0 \in J$, $a'(a x_0) = (a'a) x_0 \in J$.)

$\Rightarrow J$ is a proper ideal $\overset{17.3}{\Rightarrow} \exists$ maximal ideal M s.t. $J \subset M$.

$\overset{a)}{\Rightarrow} \exists h \in \Delta$ s.t. $\ker h = M$. Since $x_0 = e x_0 \in J \subset M \Rightarrow h(x_0) = 0$. ■

"d)" If x_0 not invertible $\Rightarrow x_0 \in J := \{a x_0 \mid a \in \mathcal{C}\}$ = proper ideal.

If x_0 invertible and J = proper ideal $\overset{17.2.a)}{\Rightarrow} x_0 \notin J$. ■

"e)" Suppose $x_0 \in \mathcal{C}$. Then $\lambda \in \sigma(x_0) \Leftrightarrow \lambda e - x_0$ not invertible $\overset{c)}{\Leftrightarrow} \exists h \in \Delta$ s.t. $h(\lambda e - x_0) = 0$. But if $h \in \Delta$, then $h(e) = 1$ and $h(\lambda e - x_0) = \lambda h(e) - h(x_0) = \lambda - h(x_0)$.

Thus $\lambda \in \sigma(x_0) \Leftrightarrow \exists h \in \Delta$ s.t. $\lambda = h(x_0)$ □

17.6. Definition: Assume $\mathcal{C}$ is a commutative Banach algebra, and let Δ collect its complex homomorphisms. For $x \in \mathcal{C}$ define its Gelfand transform as the map

$$\hat{x} : \Delta \to \mathbb{C} \quad , \quad \hat{x}(h) := h(x) \quad \forall h \in \Delta.$$

Set $\hat{\mathcal{C}} := \{\hat{x} \mid x \in \mathcal{C}\}$. The $\hat{\mathcal{C}}$-weak topology on Δ is called the Gelfand topology, and Δ endowed with this topology is called the maximal ideal space of $\mathcal{C}$.

* By definition, every $\hat{x}$ is continuous, and thus $\hat{\mathcal{C}} \subset C(\Delta) := \{f : \Delta \to \mathbb{C} \mid f \text{ is continuous}\}$.

* The name for Δ is explained by 17.5. by which $h \in \Delta, M = \ker h \Leftrightarrow M$ is a maximal ideal of $\mathcal{C}$

17.7. Definition: Assume $\mathcal{C}$ is a commutative Banach algebra. The radical of $\mathcal{C} := \bigcap\limits_{M \text{ is a maximal ideal}} M$ is denoted by $\operatorname{rad} \mathcal{C}$.

* Every ideal is a vector space and thus contains zero. On the other hand, $\{0\}$ is always a proper ideal of $\mathcal{C}$ $\overset{17.3.}{\Longrightarrow}$ $\exists$ maximal ideal $\Rightarrow$ $0 \in \operatorname{rad} \mathcal{C}$. If $\operatorname{rad} \mathcal{C} = \{0\}$ the algebra $\mathcal{C}$ is called <u>semisimple</u>. (Proves also that $\Delta \neq \emptyset$.)

<u>17.8. Theorem:</u> Let Δ be the maximal ideal space of a commutative Banach algebra $\mathcal{C}$.

a) Δ is a compact Hausdorff space, and $C(\Delta)$ is a Banach algebra, with norm $\|f\|_\infty := \sup_{h\in\Delta} |f(h)|$.

b) The map $x \mapsto \hat{x}$ is a homomorphism of $\mathcal{C}$ onto a subalgebra $\hat{\mathcal{C}}$ of $C(\Delta)$, and its kernel is $\operatorname{rad}\mathcal{C}$. It is an isomorphism <u>iff</u> $\mathcal{C}$ is semisimple.

c) For any $x \in \mathcal{C}$, the range of $\hat{x}$ is equal to $\sigma(x)$. In addition, $\|\hat{x}\|_\infty = r_\sigma(x) \leq \|x\|$, and $x \in \operatorname{rad}\mathcal{C}$ if and only if $r_\sigma(x) = 0$.

<u>Proof:</u> "b) & c)" Suppose $x, y \in \mathcal{C}$, $\alpha \in \mathbb{C}$, and $h \in \Delta$. Then $(\alpha x)^\wedge(h) = h(\alpha x) = \alpha h(x) = \alpha \hat{x}(h) = (\alpha\hat{x})(h)$, $(x+y)^\wedge(h) = h(x+y) = h(x) + h(y) = \hat{x}(h) + \hat{y}(h) = (\hat{x}+\hat{y})(h)$, and $(xy)^\wedge(h) = h(xy) = h(x)h(y) = \hat{x}(h)\hat{y}(h) = (\hat{x}\hat{y})(h)$. $\Rightarrow$ $(\alpha x)^\wedge = \alpha\hat{x}$, $(x+y)^\wedge = \hat{x}+\hat{y}$, and $(xy)^\wedge = \hat{x}\hat{y}$. Thus $\Phi: \mathcal{C} \to C(\Delta)$ def. by $\Phi(x) := \hat{x}$ is a homomorphism.
$\Rightarrow$ $\hat{\mathcal{C}} = \Phi(\mathcal{C})$ is a subalgebra. (Proof as on p. 136.)
Also $x \in \ker\Phi \Leftrightarrow \hat{x}(h) = 0 \ \forall h \in \Delta \Leftrightarrow h(x) = 0 \ \forall h \in \Delta$
$\Leftrightarrow x \in \ker h \ \forall h \in \Delta \overset{17.5.}{\Longleftrightarrow} x \in M \ \forall M = \text{maximal ideal} \Leftrightarrow x \in \operatorname{rad}\mathcal{C}$.
Thus Φ injective $\Leftrightarrow$ $\ker\Phi = \{0\}$ $\Leftrightarrow$ $\mathcal{C}$ is semisimple.
This proves "b)".

By 17.5.e), $\lambda \in \sigma(x) \Leftrightarrow \exists h \in \Delta$ s.t. $\lambda = h(x)$
$\Leftrightarrow \exists h \in \Delta$ s.t. $\lambda = \hat{x}(h) \Leftrightarrow \lambda \in \hat{x}(\Delta)$. Thus $\hat{x}(\Delta) = \sigma(x)$.
$\Rightarrow \|\hat{x}\|_\infty = \sup_{h\in\Delta} |\hat{x}(h)| = \sup_{\lambda\in\sigma(x)} |\lambda| = r_\sigma(x) \leq \|x\|$. Therefore

$x \in \operatorname{rad}\mathcal{C} \Leftrightarrow \hat{x} = 0 \Leftrightarrow 0 = \|\hat{x}\|_\infty = r_\sigma(x)$. This proves "c)".

"a)" Let $\mathcal{C}^*$ denote the dual of $\mathcal{C}$. By Ex. 8,2. the set $K := \{\Lambda \in \mathcal{C}^* \mid \|\Lambda\| \leq 1\}$ is weak*-compact.
(Let $V_0 = B(0,1) \subset \mathcal{C}$. Then $\Lambda \in K \Leftrightarrow |\Lambda x| \leq 1 \ \forall \|x\| \leq 1$
$\Leftrightarrow |\Lambda x| \leq 1 \ \forall x \in V_0 \Leftrightarrow \Lambda \in$ Polar of V_0 = weak*-compact by Banach-Alaoglu.)

$\|x\| = 1 \Rightarrow \forall \varepsilon > 0$:
$|\Lambda x| = \frac{1}{1-\varepsilon} |\Lambda(\underbrace{(1-\varepsilon)x}_{\in V_0})| \leq \frac{1}{1-\varepsilon} \Rightarrow |\Lambda x| \leq 1$.

If $h \in \Delta \overset{16.8.}{\Rightarrow} h \in \mathcal{C}^*$ and $\|h\| \leq 1 \Rightarrow h \in K$. Thus $\Delta \subset K$.
Let τ denote the Gelfand topology on Δ and τ' the relative topology induced by the weak*-topology on Δ.
Since $\hat{\mathcal{C}}$ separates points on Δ ($h_1 \neq h_2 \Rightarrow \exists x \in \mathcal{C}$ s.t. $h_1(x) \neq h_2(x) \Rightarrow \hat{x}(h_1) \neq \hat{x}(h_2)$ with $\hat{x} \in \hat{\mathcal{C}}$), and $\mathbb{C}$ is Hausdorff $\Rightarrow \tau$ is Hausdorff by Lemma 13.2.
We will soon prove that Δ is weak*-closed $\Rightarrow \Delta$ is weak* compact (as K is) $\Rightarrow \Delta$ is compact under τ'. On the other hand, $\hat{x} = f_x|_\Delta$ where $f_x : \mathcal{C}^* \to \mathbb{C}$ is weak*-contin.
$\Rightarrow \hat{x}$ is τ'-continuous (see p. 54) $\overset{\text{by def.}}{\Rightarrow} \tau \subset \tau'$, where τ is Hausdorff and τ' compact $\overset{\text{Ex. 8.3a)}}{\Rightarrow} \tau = \tau'$.
$\Rightarrow \Delta$ is compact under τ. By Ex. 10.2. $\Rightarrow C(\Delta)$ is a Banach algebra.

Thus to complete the proof of "a)", it suffices to show that Δ is weak*-closed. Consider $\Lambda_0 \in \bar{\Delta}^{\mathcal{T}_{w*}} \Rightarrow \Lambda_0 \in \mathcal{C}^* \Rightarrow \Lambda_0$ is linear map $\mathcal{C} \to \mathbb{C}$. Suppose $x_1, x_2 \in \mathcal{C}$, and denote $x_3 = x_1 x_2$, $x_4 = e$.
For $\varepsilon > 0$, define $V := \{\Lambda \in \mathcal{C}^* \mid |\Lambda x_i - \Lambda_0 x_i| < \varepsilon \ \forall i \in \{1,2,3,4\}\}$
$\Rightarrow V \in \mathcal{T}_{w*}$ and $\Lambda_0 \in V \Rightarrow \exists h \in \Delta \cap V$. Then $h \in \Delta \Rightarrow h(e) = 1$
$\Rightarrow |1 - \Lambda_0 e| = |h(e) - \Lambda_0 e| < \varepsilon$. Thus $\Lambda_0 e = 1 \Rightarrow \Lambda_0 \neq 0$.
Also $\Lambda_0(x_1x_2) - (\Lambda_0 x_1)(\Lambda_0 x_2) = \Lambda_0(x_1x_2) - h(x_1x_2) + h(x_1)h(x_2) - ()()$
$= \Lambda_0(x_1x_2) - h(x_1x_2) + (h(x_1) - \Lambda_0 x_1)h(x_2) + \Lambda_0 x_1 \cdot (h(x_2) - \Lambda_0 x_2)$
$\Rightarrow |\Lambda_0(x_1x_2) - (\Lambda_0 x_1)(\Lambda_0 x_2)| < \varepsilon + |h(x_2)|\varepsilon + \varepsilon|\Lambda_0 x_1 - h(x_1) + h(x_1)|$
$\leq \varepsilon(1 + \|x_2\| + \|x_1\| + \varepsilon)$, by 16.8. As ε is arbitrary
$\Rightarrow \Lambda_0(x_1x_2) = (\Lambda_0 x_1)(\Lambda_0 x_2)$. $\therefore \Lambda_0 \in \Delta$ $\therefore \Delta$ = weak*-closed $\square$

* Here $\hat{\mathcal{C}}$ may or may not be closed in $C(\Delta)$, i.e., it does not need to be a Banach algebra.

17.9. Theorem: Assume $\mathcal{C}$ and $\mathcal{S}$ are commutative Banach algebras, and that $\mathcal{S}$ is semisimple. If $\Phi : \mathcal{C} \to \mathcal{S}$ is a homomorphism, then Φ is continuous.

Proof: By the closed graph theorem (11.7. and 11.5.) it suffices to show that whenever $x_n \to x$ in $\mathcal{C}$ is such that $\Phi(x_n) \to y$ in $\mathcal{S}$ we must have $y = \Phi(x)$.

Let $\Delta_{\mathcal{C}}$ and $\Delta_{\mathcal{S}}$ be the maximal ideal spaces. Suppose $h \in \Delta_{\mathcal{S}} \overset{16.8.}{\Rightarrow} h : \mathcal{S} \to \mathbb{C}$ is continuous, But then $\varphi := h \circ \Phi : \mathcal{C} \to \mathbb{C}$ is a homomorphism. If $\varphi \not\equiv 0$, then $\varphi \in \Delta_{\mathcal{S}} \overset{16.8.}{\Rightarrow}$ it is also continuous, and if $\varphi \equiv 0$, it is obviously continuous. Therefore, $h(y) = \lim_{n\to\infty} h(\Phi(x_n)) = \lim_{n\to\infty} \varphi(x_n) = \varphi(x) = h(\Phi(x))$

$\Rightarrow h(y - \Phi(x)) = 0 \ \forall h \in \Delta_{\mathcal{S}} \Rightarrow y - \Phi(x) \in M \ \forall M =$ max. ideal in $\mathcal{S}$
$\Rightarrow y - \Phi(x) \in \operatorname{rad} \mathcal{S} = \{0\} \Rightarrow y = \Phi(x) \ \therefore \ \Phi$ is contin. $\square$

<u>17.10. Corollaries</u>: Suppose $\mathcal{S}$ and $\mathcal{S}_2$ are semisimple commutative Banach algebras.

a) If $\Phi : \mathcal{S} \to \mathcal{S}_2$ is an isomorphism, it is also a homeomorphism.
b) Any automorphism of $\mathcal{S}$ is a homeomorphism. (i.e. algebra determines topology on $\mathcal{S}$.)

<u>Proof.</u> "a)" $\Phi^{-1} : \mathcal{S}_2 \to \mathcal{S}$ is also a homomorphism (easy check.) Thus by 17.9. $\Rightarrow \Phi$ and Φ^{-1} are continuous.
"b)" Follows from a) $\square$