and $x' \in B(x,\delta)$, $\delta := \frac{\varepsilon}{4\|x^{-1}\|^2} > 0$, we have $\|\Phi(x') - \Phi(x)\| < \varepsilon$.

Therefore, Φ is continuous $\Rightarrow \Phi^{-1} = \Phi$ is contin.
$\therefore$ Φ is a homeomorphism. $\square$

16.14. Theorem: Let A be a Banach algebra and $x \in A$. Then:

(a) The spectrum $\sigma(x)$ of x is compact and nonempty $\subset \mathbb{C}$.
(b) The resolvent $\lambda \mapsto R_\lambda(x)$ is a strongly holomorphic map $\rho(x) \to A$.
(c) The spectral radius $r_\sigma(x)$ of x satisfies

$$r_\sigma(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|x^n\|^{\frac{1}{n}}$$

In particular, $0 \leq r_\sigma(x) \leq \|x\| < \infty$.

Proof: If $\lambda \in \mathbb{C}$, $|\lambda| > \|x\|$, then $\|\lambda^{-1}x\| < 1 \overset{16.7.a}{\Rightarrow} e - \lambda^{-1}x \in G(A)$
$\Rightarrow \lambda e - x = \lambda(e - \lambda^{-1}x) \in G(A) \Rightarrow \lambda \notin \sigma(x)$. This proves that $\sigma(x) \subset \overline{B}(0, \|x\|)$. $\Rightarrow \rho(x) \neq \emptyset$.

Consider then $g: \mathbb{C} \to A$ defined by $g(\lambda) = \lambda e - x$. As $g(\lambda') - g(\lambda) = (\lambda' - \lambda)e \Rightarrow \|g(\lambda') - g(\lambda)\| = |\lambda' - \lambda|$, g is continuous $\Rightarrow g|_{\rho(x)}: \rho(x) \to G(A)$ is continuous
$\overset{16.11.}{\Rightarrow}$ $\Phi \circ (g|_{\rho(x)}): \rho(x) \to G(A)$ is continuous. Since for $\lambda \in \rho(x)$ $\Phi(g(\lambda)) = (\lambda e - x)^{-1} = R_\lambda(x)$, the resolvent is contin. In addition, $\rho(x) = g^{\leftarrow}(G(A))$ is open as $G(A)$ is open (16.11.).
$\Rightarrow \sigma(x) = \mathbb{C} \setminus \rho(x)$ is closed. $\therefore$ $\sigma(x)$ is compact.

Consider then the resolvent map $f: \rho(x) \to A$ def. by $f(\lambda) := R_\lambda(x) \in G(A)$. Consider $\lambda \in \rho(x)$, and $\mu \in \mathbb{C}$ s.t. $|\mu - \lambda| < \frac{1}{2\|(\lambda e - x)^{-1}\|}$. Set $y := (\mu - \lambda)e$
$\Rightarrow \|y\| = |\mu - \lambda|$, $\lambda e - x \in G(A)$.
Thus by Lemma 16.10. we then have $\lambda e - x + (\mu - \lambda)e$
$= \mu e - x \in G(A) \Rightarrow \mu \in \rho(x)$, and also

$$\|(\mu e - x)^{-1} - (\lambda e - x)^{-1} + (\lambda e - x)^{-1}(\mu - \lambda)e(\lambda e - x)^{-1}\|$$
$$= \|f(\mu) - f(\lambda) + (\mu - \lambda)f(\lambda)^2\| \leq 2|\mu - \lambda|^2 \|f(\lambda)\|^3.$$

Thus $\left\| \frac{f(\mu) - f(\lambda)}{\mu - \lambda} + f(\lambda)^2 \right\| \leq 2|\mu - \lambda| \, \|f(\lambda)\|^3 \xrightarrow{\mu \to \lambda} 0$

$\Rightarrow f$ is strongly holomorphic on $\rho(x)$, and $f'(\lambda) = -f(\lambda)^2$ $\forall \lambda \in \rho(x)$.
This proves "b)".

We proved that if $|\lambda| > \|x\|$, then $\lambda \in \rho(x)$. In addition, by 16.7.a) then

$$(e - \lambda^{-1}x)^{-1} = \sum_{n=0}^{\infty} (\lambda^{-1}x)^n$$ and the series is absolutely convergent. $\Rightarrow$ $f(\lambda) = (\lambda e - x)^{-1} = \lambda^{-1}(e - \lambda^{-1}x)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} x^n$.

For $r > 0$, let $\gamma_r(t) := re^{it}$, $t \in [0, 2\pi]$ = pos. oriented circle at 0 with radius r. For $n = 0, 1, \ldots$, set
$g_n(t) := \gamma_r'(t)\,(\gamma_r(t))^{-(n+1)} x^n = i r^{-n} e^{-int} x^n$.
$\Rightarrow$ $g_n : [0, 2\pi] \to A$ is contin. and $\|g_n(t)\| \le r^{-n}\|x\|^n$.

Thus if $r > \|x\|$, we have $\sum_{n=1}^{\infty} \int_0^{2\pi} dt\, \|g_n(t)\| \le 2\pi \sum_{n=1}^{\infty} \left(\frac{\|x\|}{r}\right)^n < \infty$.

By Ex. 10.5., this implies that for $r > \|x\|$, $n_0 \in \{0, 1, 2, \ldots\}$,

$$\frac{1}{2\pi i} \oint_{\gamma_r} d\lambda\, \lambda^{n_0} f(\lambda) = \frac{1}{2\pi i} \int_0^{2\pi} dt\, \gamma_r(t)^{n_0} \sum_{n=0}^{\infty} g_n(t)$$
$$= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_0^{2\pi} dt\, \gamma_r(t)^{n_0} g_n(t)$$
$$\overset{\text{apply } \Lambda \in A^*}{=} \sum_{n=0}^{\infty} \underbrace{\left(\int_0^{2\pi} \frac{dt}{2\pi} r^{n_0 - n} e^{i(n_0 - n)t} \right)}_{= 0 \text{ if } n \ne n_0, \text{ and } = 1 \text{ if } n = n_0} x^n = x^{n_0}.$$

Since f is strongly holom. also $\lambda^{n_0} f(\lambda)$ is strongly holom. on $\rho(x)$. (Proof $\Lambda \in A^* \Rightarrow \Lambda \circ f$ is holom. $\Rightarrow \lambda \mapsto \lambda^{n_0} \Lambda(f(\lambda)) = \Lambda(\lambda^{n_0} f(\lambda))$ is holom. Thus $\lambda^{n_0} f$ is weakly holom. $\Rightarrow$ strongly holom. (by 15.8.)) Thus can apply Cauchy.
If $\sigma(x) = \emptyset$ $\Rightarrow$ $\rho(x) = \mathbb{C}$ $\overset{\text{Cauchy}}{\Rightarrow}$ $\frac{1}{2\pi i}\oint d\lambda\, f(\lambda) = 0 = x^0 = e$ ↯
(Since $e \ne 0$.) Thus $\sigma(x) \ne \emptyset$ and we have proven a). ⊛
We proved above that $\forall r > \|x\|$ it holds that

(∗) $\frac{1}{2\pi i} \oint_{\gamma_r} d\lambda\, \lambda^{n_0} f(\lambda) = x^{n_0}$. Since $\lambda^{n_0} f(\lambda)$ is analytic

in $\rho(x)$ and (by def.) $\overline{B(0, r_\sigma(x))}^c \subset \rho(x)$, this holds for every $r > r_\sigma(x)$ (≥ 0). (By Cauchy, 15.8.b)
Denote $M(r) := \sup_{\lambda \in R(\gamma_r)} \|f(\lambda)\|$. As $R(\gamma_r)$ is compact

and f is continuous $\Rightarrow$ $M(r) < \infty$. But then by (∗) and 15.3.b)

⊛ and that $0 \le r_\sigma(x) \le \|x\|$.

$$\Rightarrow \|x^{n_0}\| \le \frac{1}{2\pi}\int_0^{2\pi} dt\; r^{n_0+1} M(r) = r^{n_0+1}M(r)$$

$$\Rightarrow \|x^{n_0}\|^{\frac{1}{n_0}} \le r\,(rM(r))^{\frac{1}{n_0}} \xrightarrow{n_0\to\infty} r$$

$$\Rightarrow \limsup_{n\to\infty} \|x^n\|^{\frac{1}{n}} \le r \quad \forall r > r_\sigma(x)$$

$$\Rightarrow \limsup_{n\to\infty} \|x^n\|^{\frac{1}{n}} \le r_\sigma(x). \quad \left(\limsup_{n\to\infty} c_n := \inf_{n_0\ge 1}\left(\sup_{n\ge n_0} c_n\right).\right)$$

Suppose then $\lambda \in \sigma(x)$. Now $\forall n_0 \in \mathbb{N}_+$

$$(\lambda e - x)\sum_{n=0}^{n_0-1} \lambda^{n_0-1-n}x^n = \sum_{n=0}^{n_0-1}\lambda^{n_0-n}x^n - \sum_{n'=1}^{n_0}\lambda^{n_0-n'}x^{n'} \quad (n'=n+1)$$

$$= \lambda^{n_0}e - x^{n_0} = \left(\sum_{n=0}^{n_0-1}\lambda^{n_0-1-n}x^n\right)(\lambda e - x)$$

Since $\lambda e - x \notin G(\mathcal{A})$, the Lemma below shows that $\lambda^{n_0}e - x^{n_0} \notin G(\mathcal{A})$. $\Rightarrow \lambda^{n_0} \in \sigma(x^{n_0}) \Rightarrow |\lambda^{n_0}| \le r_\sigma(x^{n_0}) \le \|x^{n_0}\|$.
$\Rightarrow |\lambda| \le \|x^{n_0}\|^{\frac{1}{n_0}}$. Therefore, $r_\sigma(x) \le \|x^n\|^{\frac{1}{n}}\ \forall n \in \mathbb{N}_+$.
Denote $c_n := \|x^n\|^{\frac{1}{n}}$ and $r = r_\sigma(x)$. We have now proven that $\limsup_{n\to\infty} c_n \le r$ and $r \le c_n\ \forall n \Rightarrow r \le \liminf_{n\to\infty} c_n$

$$\Rightarrow \liminf c_n \le \limsup c_n \le r \le \liminf c_n \le \limsup c_n \Rightarrow \limsup = \liminf$$

$$\Rightarrow \exists \lim_{n\to\infty} c_n = r \text{ and } r \le \inf_{n\ge 1} c_n \le \liminf_{n\to\infty} c_n = r \Rightarrow r = \inf_{n\ge 1} c_n \quad \square$$

16.15. Lemma: If $x \in G(\mathcal{A})$ and $a, b \in \mathcal{A}$ are such that $x = ab = ba$, then $a, b \in G(\mathcal{A})$.

Proof. $e = x^{-1}(ba) = (x^{-1}b)a$ and $e = (ab)x^{-1} = a(bx^{-1})$

$$\Rightarrow x^{-1}b = (x^{-1}b)e = (x^{-1}b)(a(bx^{-1})) = ((x^{-1}b)a)(bx^{-1})$$

$$= e(bx^{-1}) = bx^{-1} \Rightarrow a(bx^{-1}) = e = (bx^{-1})a \Rightarrow a \in G(\mathcal{A})$$

$$\Rightarrow b = a^{-1}x \in G(\mathcal{A}) \quad \square$$

16.16 Theorem (Gelfand-Mazur)

If $\mathcal{A}$ is a Banach algebra in which every nonzero element is invertible, then $\mathcal{A} \cong \mathbb{C}$ (with isometric isomorphism).

<u>Proof</u>. Suppose $x \in A$, $\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{C}$.
$\Rightarrow$ either $\lambda_1 e - x$ or $\lambda_2 e - x \in G(A)$. (else both are zero ↯)
By 16.14, $\sigma(x) \neq \emptyset$, and thus $\exists! \lambda(x) \in \mathbb{C}$ s.t. $\sigma(x) = \{\lambda(x)\}$.
But by assumption, $\lambda(x)e - x \notin G(A) \Rightarrow \lambda(x)e - x = 0 \Rightarrow x = \lambda(x)e$.
$\Rightarrow \|x\| = |\lambda(x)| \;\forall x \in A$. In addition, $(\alpha\lambda(x) + \beta\lambda(y))e$
$= \alpha(\lambda(x)e) + \beta(\lambda(y)e) = \alpha x + \beta y = \lambda(\alpha x + \beta y)e$
$\Rightarrow \lambda(\alpha x + \beta y) = \alpha\lambda(x) + \beta\lambda(y)$, since $e \neq 0 \Rightarrow ze \neq 0 \;\forall z \neq 0$.
Thus λ is linear. Also $(\lambda(x)\lambda(y))e = \lambda(x)(\lambda(y)e) = \lambda(x)y$
$= \lambda(x)(ey) = (\lambda(x)e)y = xy = \lambda(xy)e \Rightarrow \lambda(xy) = \lambda(x)\lambda(y)$.
$\therefore \lambda$ is homomorphism. If $\lambda(x) = \lambda(y) \Rightarrow x = \lambda(x)e = \lambda(y)e = y$.
Also $\alpha \in \mathbb{C} \Rightarrow \alpha e \in A \Rightarrow \lambda(\alpha e) = \alpha\lambda(e) = \alpha$ by 16.6.
($\lambda \equiv 0 \Rightarrow \forall x \in A : x = \lambda(x)e = 0e = 0 \Rightarrow e = 0$ ↯). This proves that λ is bijective. $\therefore \lambda : A \to \mathbb{C}$ is an isometric isomorphism. □

* Note that the definition of the spectrum <u>depends on the algebra</u>: If A is a Banach subalgebra of a Banach algebra A', then $G(A) \subset G(A')$, and thus $\sigma_A(x) \supset \sigma_{A'}(x) \;\forall x \in A$, but the inclusion can be strict. Nevertheless, by 16.14, $r_{\sigma_A}(x) = \inf_{n \geq 1} \|x^n\|_A^{1/n} = \inf_{n \geq 1} \|x^n\|_{A'}^{1/n} = r_{\sigma_{A'}}(x)$. The following theorem restricts the possibilities further:

<u>16.17. Theorem</u>: Suppose A' is a Banach algebra, and $A \subset A'$ is its subalgebra. Assume further that $e \in A$ and A is closed in A'. Then:

a) A is a Banach algebra.
b) $G(A)$ is a union of connected components of $A \cap G(A')$.
c) For any $x \in A$, we can write $\sigma_A(x) = \sigma_{A'}(x) \cup \left(\bigcup_{i \in I} C_i(x)\right)$, where I is an index set, and every $C_i(x)$ is a <u>bounded connected component</u> of $\rho_{A'}(x) := \mathbb{C} \setminus \sigma_{A'}(x)$. In addition, the boundary of $\sigma_A(x)$ is a <u>nonempty</u> subset of $\sigma_{A'}(x)$.

d) If $\rho_{A'}(x)$ is connected (i.e. $\sigma_{A'}(x)$ does not separate $\mathbb{C}$), then $\sigma_A(x) = \sigma_{A'}(x)$.

We begin the proof with two Lemmas:

16.18. Lemma: Suppose V, W are open in $\mathfrak{X}$. If $V \subset W$ and W contains no boundary point of V, then V is a union of components of W.

Proof. Let Ω be a connected component of W, and suppose $\Omega \cap V \neq \emptyset$. Set $U := \mathfrak{X} \setminus \overline{V}$, which is open in $\mathfrak{X}$.
Now $\partial V :=$ collection of boundary points of $V = \overline{V} \cap \overline{V^c}$, $V^c = \mathfrak{X} \setminus V$.
By assumption, $W \cap (\partial V) = \emptyset$. Thus if $x \in W \Rightarrow x \in (\partial V)^c$
$= \overline{V}^c \cup (\overline{V^c})^c = U \cup (\overline{V^c})^c$. Here $V^c \subset \overline{V^c} \Rightarrow V \supset \overline{V^c}^c$, and thus $W \subset U \cup V \Rightarrow \Omega \subset U \cup V \Rightarrow \Omega = (\Omega \cap U) \cup (\Omega \cap V)$.
The sets $\Omega \cap U$ and $\Omega \cap V$ are disjoint and open in Ω.
Since Ω is connected and $\Omega \cap V \neq \emptyset$, this implies $\Omega \cap U = \emptyset$.
$\therefore \Omega \subset V$. $\square$

16.19. Lemma: Suppose $\mathcal{A}$ is a Banach algebra. Consider a sequence $x_n \in G(\mathcal{A})$, $n \in \mathbb{N}_+$, which converges to a boundary point x of $G(\mathcal{A})$ in $\mathcal{A}$. Then $\|x_n^{-1}\| \to \infty$ as $n \to \infty$.

Proof: Suppose $\|x_n^{-1}\| \not\to \infty$. Then $\exists M > 0$ s.t. $\forall n\ \exists k(n) \geq n$ for which $\|x_{k(n)}^{-1}\| < M$. Since $x_{k(n)} \xrightarrow{n \to \infty} x$, we can find $n' \in \mathbb{N}_+$ s.t. $\|x - x_{k(n')}\| < \frac{1}{M}$. Denote $n_0 := k(n')$.
Then $\|e - x_{n_0}^{-1} x\| = \|x_{n_0}^{-1}(x_{n_0} - x)\| \leq \|x_{n_0}^{-1}\| \|x_{n_0} - x\| < M \cdot \frac{1}{M} = 1$.
$\overset{16.7.a)}{\Rightarrow} e - (e - x_{n_0}^{-1} x) = x_{n_0}^{-1} x \in G(\mathcal{A})$. $\overset{16.4.}{\Rightarrow} x = x_{n_0}(x_{n_0}^{-1} x) \in G(\mathcal{A})$.
But as $x \in \partial(G(\mathcal{A})) \Rightarrow x \in \overline{G(\mathcal{A})^c} = G(\mathcal{A})^c$ since $G(\mathcal{A})$ is open in $\mathcal{A}$ (by 16.11.) $\lightning$ Thus $\|x_n^{-1}\| \to \infty$ $\square$

Proof of 16.17: "a)" Since $\mathcal{A} \subset \mathcal{A}'$ is a closed subspace $\overset{2.12.a)}{\Rightarrow}$ $\mathcal{A}$ is a Banach space. Since $e_{\mathcal{A}'} \in \mathcal{A}$
$\Rightarrow e_{\mathcal{A}'} x = x = x e_{\mathcal{A}'}$ $\forall x \in \mathcal{A}$ and $e_{\mathcal{A}} = e_{\mathcal{A}'} \Rightarrow \|e_{\mathcal{A}}\| = 1$.
Also $\|xy\| \leq \|x\| \|y\|$ $\forall x, y \in \mathcal{A}$. $\therefore$ $\mathcal{A}$ is a Banach algebra. $\blacksquare$

"b)" As mentioned above, obviously now $G(\mathcal{A}) \subset G(\mathcal{A}')$. By 16.11. both $G(\mathcal{A})$ and $\mathcal{A} \cap G(\mathcal{A}')$ are open in $\mathcal{A}$. Let y be a boundary point of $G(\mathcal{A})$ in $\mathcal{A}$. Since $y \in \overline{G(\mathcal{A})}$, $\exists y_n \in G(\mathcal{A})$, $n \in \mathbb{N}_+$, s.t. $y_n \to y$, and then by 16.19. we have $\|y_n^{-1}\| \to \infty$.
If now $y \in G(\mathcal{A}')$, we would have $y_n \to y$ in $G(\mathcal{A}')$, and as inversion is continuous (16.11.) $\Rightarrow y_n^{-1} \to y^{-1}$ in $G(\mathcal{A}')$.

$\Rightarrow \|y_n^{-1}\| \xrightarrow{n\to\infty} \|y^{-1}\| < \infty$ ↯. Thus $y \notin G(A')$.
As $G(A) \subset A \cap G(A')$, we can conclude from 16.18. that $G(A)$ is a union of components of $A \cap G(A')$ ◼

"c)" Denote $\sigma(x) := \sigma_A(x)$, $\sigma'(x) := \sigma_{A'}(x)$, $\rho(x) = \mathbb{C} \setminus \sigma(x)$ and $\rho'(x) = \mathbb{C} \setminus \sigma'(x)$. By 16.14. a), $\rho(x)$ and $\rho'(x)$ are open and $\rho(x) \subset \rho'(x)$ since $\lambda \in \rho(x) \Rightarrow \lambda e - x \in G(A) \subset G(A') \Rightarrow \lambda \in \rho'(x)$.
Let λ_0 be a boundary point of $\rho(x)$. $\Rightarrow \lambda_0 \in \overline{\rho(x)} \cap \rho(x)^c$
$\Rightarrow \lambda_0 e - x \notin G(A)$ and $\exists \mu_n \in \rho(x)$ s.t. $\mu_n \to \lambda_0$. But then $\mu_n e - x \in G(A)$ and $\mu_n e - x \to \lambda_0 e - x$ in A. $\Rightarrow \lambda_0 e - x \in \overline{G(A)} \cap G(A)^c$
$= \partial(G(A))$. But in "b)" it was shown that $\partial(G(A)) \cap G(A') = \emptyset$.
$\Rightarrow \lambda_0 e - x \notin G(A') \Rightarrow \lambda_0 \notin \rho'(x)$. Thus by 16.18, $\rho(x)$ is a union of components of $\rho'(x)$. Since the components partition $\rho'(x) \Rightarrow$ adding $\sigma'(x)$ partitions $\mathbb{C}$. Let I index those components of $\rho'(x)$ which intersect $\sigma(x)$. Thus if $i \in I$
$\Rightarrow C_i(x) \cap \rho(x) = \emptyset$ (as else $C_i(x) \subset \rho(x) \Rightarrow C_i(x) \cap \sigma(x) = \emptyset$ ↯)
$\Rightarrow C_i(x) \subset \sigma(x)$. Since $\sigma'(x) \subset \sigma(x)$, we can conclude that $\sigma(x) = \sigma'(x) \cup \left(\bigcup_{i \in I} C_i(x) \right)$, as claimed. By 16.14. a), $\sigma(x)$

is bounded $\Rightarrow$ every $C_i(x) \subset \sigma(x)$ is bounded. Since $\partial(\sigma(x)) = \partial(\sigma(x)^c) = \partial(\rho(x))$, we can also conclude that $\partial(\sigma(x)) \subset \rho'(x)^c = \sigma'(x)$. If $\partial\sigma = \emptyset \Rightarrow \overline{\sigma} \cap \overline{\sigma^c} = \emptyset \overset{\sigma \text{ closed}}{\Rightarrow} \sigma \cap \overline{\sigma^c} = \emptyset$
$\Rightarrow \overline{\sigma^c} \subset \sigma^c \Rightarrow \overline{\sigma^c} = \sigma^c \Rightarrow \sigma$ is closed and open $\overset{\mathbb{C} \text{ connected}}{\Rightarrow} \sigma = \emptyset$ or $\sigma = \mathbb{C}$ ↯ □

"d)" If $\rho'(x)$ is connected $\Rightarrow$ it has no bounded components ($\rho'(x)^c = \sigma'(x)$ is bounded) $\overset{c)}{\Rightarrow} \sigma(x) = \sigma'(x)$ □

<u>16.20. Theorem</u>: If A is a Banach algebra and $\exists M \geq 0$ such that $\|x\| \cdot \|y\| \leq M \|xy\| \;\; \forall x, y \in A$, then $A \cong \mathbb{C}$ (isometrically isomorphic).

<u>Proof:</u> Let $y \in \partial(G(A)) \Rightarrow \exists y_n \in G(A)$, $n \in \mathbb{N}_+$, s.t. $y_n \to y$
$\overset{16.15.}{\Rightarrow} \|y_n^{-1}\| \to \infty$. But $\|y_n\| \|y_n^{-1}\| \leq M \|y_n y_n^{-1}\| = M$
$\Rightarrow \|y_n\| \xrightarrow{n\to\infty} 0 \Rightarrow y_n \to 0 \Rightarrow y = 0$. $\therefore \partial(G(A)) \subset \{0\}$.
If $x \in A$ and $\lambda \in \partial(\sigma(x)) \Rightarrow \lambda e - x \in \partial(G(A))$ as in the proof of "c)" above $\Rightarrow \lambda e - x = 0 \Rightarrow x = \lambda e$. By 16.17., $\partial(\sigma(x)) \neq \emptyset$ and thus $\forall x \in A \; \exists \lambda(x) \in \mathbb{C}$ s.t. $x = \lambda(x) e$. As in the proof of 16.16., the map $x \mapsto \lambda(x)$ must then be an isometric isomorphism $A \to \mathbb{C}$ □