16. Banach algebras and spectrum

16.1. Definition: $\mathbb{A}$ is an algebra (over $\mathbb{k}$) if

a) $\mathbb{A}$ is a vector space with multiplication $\mathbb{A}\times\mathbb{A}\to\mathbb{A}$
b) $x(yz) = (xy)z \quad \forall x,y,z \in \mathbb{A}$
c) $(x+y)z = xz + yz$ and $x(y+z) = xy + xz \quad \forall x,y,z \in \mathbb{A}$
d) $\alpha(xy) = (\alpha x)y = x(\alpha y). \quad \forall x,y \in \mathbb{A},\ \alpha \in \mathbb{k}.$

Here $\mathbb{A}$ is called a Banach algebra if

a) $\mathbb{k} = \mathbb{C}$
b) $\mathbb{A}$ is an algebra
c) $\mathbb{A}$ is a Banach space (with norm $\|\cdot\|$)
d) $\exists e \in \mathbb{A}$ s.t. it is a unit element:
$$xe = ex = x \quad \forall x \in \mathbb{A}$$
e) $\|e\| = 1$
f) $\|xy\| \leq \|x\|\,\|y\| \quad \forall x,y \in \mathbb{A}.$

* The unit element is unique: $e' = e'e = e$, and $\neq 0$.
* If V is a complex Banach space, $\mathcal{B}(V)$ is a Banach algebra with $e = id$ = identity map $V \to V$.
(Proof: Ex. 8.1. and obvious computations.)

16.2. Proposition: If $\mathbb{A}$ is a Banach algebra, the map $(x,y) \mapsto xy$ is a continuous map $\mathbb{A}\times\mathbb{A}\to\mathbb{A}$.

Proof. $xy - x_0y_0 = x(y-y_0) + (x-x_0)y_0$. Thus if $\varepsilon > 0$ and $\|(x,y)-(x_0,y_0)\|_{\mathbb{A}\times\mathbb{A}} < \varepsilon \Rightarrow \|x-x_0\| + \|y-y_0\| < \varepsilon$

$$\Rightarrow \|xy - x_0y_0\| \leq \underbrace{\|x\|}_{\leq \|x_0\| + \|x-x_0\|}\|y-y_0\| + \|y_0\|\,\|x-x_0\|$$

$$\leq \varepsilon(\|x_0\| + \|y_0\| + \varepsilon) \xrightarrow{\varepsilon \to 0} 0. \quad \square$$

* Thus, in particular, if $x_n \to x$, $y_n \to y$ are two convergent sequences, then $x_ny_n \to xy$, $x_ny \to xy$ and $xy_n \to xy$.

16.3. Definition: Let $\mathcal{A}$ be a Banach algebra. Define $G(\mathcal{A}) := \{ x \in \mathcal{A} \mid \exists y \in \mathcal{A} \text{ s.t. } xy = e = yx \}$.
$G(\mathcal{A})$ is called the <u>group of invertible elements of $\mathcal{A}$</u>.

* If $x \in G(\mathcal{A})$, it is called <u>invertible</u> (in $\mathcal{A}$) and $y \in \mathcal{A}$ with $xy = e = yx$ is the <u>inverse</u> of x, denoted by x^{-1}. (The inverse is unique: $y = ye = y(xx^{-1}) = (yx)x^{-1} = ex^{-1} = x^{-1}$.)

16.4. Proposition: $G(\mathcal{A}) \neq \phi$ and it is a group under $(x,y) \mapsto xy$. In addition, $0 \notin G(\mathcal{A})$.

Proof. Since $ee = e \Rightarrow e \in G(\mathcal{A})$ and $e^{-1} = e$.
If $x, y \in G(\mathcal{A}) \Rightarrow \exists x^{-1}, y^{-1} \in G(\mathcal{A})$ and $(xy)(y^{-1}x^{-1}) = xex^{-1} = xx^{-1} = e = (y^{-1}x^{-1})(xy) \Rightarrow (xy)^{-1} = y^{-1}x^{-1}$ and $xy \in G(\mathcal{A})$. $\Rightarrow \exists$ identity element $= e \in G(\mathcal{A})$ and $(xy)z = x(yz)$ $\forall x, y, z \in G(\mathcal{A})$. Also if $x \in G(\mathcal{A})$ then $x^{-1}x = e = xx^{-1} \Rightarrow x^{-1} \in G(\mathcal{A})$ and $(x^{-1})^{-1} = x$.
Since $2(0_{\mathcal{A}}x) = (20_{\mathcal{A}})x = 0_{\mathcal{A}}x \Rightarrow 0_{\mathcal{A}}x = 0_{\mathcal{A}} \neq e$ $\forall x$. Thus $0_{\mathcal{A}} \notin G(\mathcal{A})$ $\square$

16.5. Definition: Assume $\mathcal{A}$ is a complex algebra. If $h: \mathcal{A} \to \mathbb{C}$ is linear, $h \neq 0$, and

$$h(xy) = h(x)h(y) \quad \forall x, y \in \mathcal{A}$$

then h is called a <u>complex homomorphism</u> on $\mathcal{A}$.

16.6. Proposition: Assume h is a complex homomorphism on a complex algebra $\mathcal{A}$ with unit e. Then $h(e) = 1$ and $h(x) \neq 0$ for every invertible $x \in \mathcal{A}$.

Proof. Since $h \neq 0 \Rightarrow \exists y \in \mathcal{A}$ s.t. $h(y) \neq 0$. As $h(y)h(e) = h(ye) = h(y) \Rightarrow h(e) = 1$. If $\exists x^{-1}$ then $h(x)h(x^{-1}) = h(xx^{-1}) = h(e) = 1 \Rightarrow h(x) \neq 0$. $\square$

16.7. Theorem: Suppose $\mathcal{A}$ is a <u>Banach algebra</u>. If $x \in \mathcal{A}$ and $\|x\| < 1$, then

(a) $e - x \in G(\mathcal{A})$, and $(e-x)^{-1} = \sum_{n=0}^{\infty} x^n$.

(b) $\|(e-x)^{-1} - e - x\| \leq \dfrac{\|x\|^2}{1 - \|x\|}$

(c) $|h(x)| < 1$ for every complex homom. h on $\mathcal{A}$.

Proof: By easy induction, $\|x^n\| \le \|x\|^n$ $\forall x \in A, n \in \mathbb{N}_+$.
Set $x^0 := e$ $\forall x \in A \Rightarrow \|x^0\| = \|e\| = 1 = \|x\|^0$. $(0^0 = 1)$
As $\|x\| < 1 \Rightarrow \sum_{n=0}^{\infty} \|x\|^n = \frac{1}{1-\|x\|} < \infty$. Since A is a Banach space $\Rightarrow \exists\, y = \sum_{n=0}^{\infty} x^n := \|\cdot\|\text{-}\lim_{N\to\infty} \sum_{n=0}^{N} x^n$.

By 16.2. we thus have $y(e-x) = \lim_{N\to\infty} \left[\left(\sum_{n=0}^{N} x^n\right)(e-x)\right]$

$= \lim_{N\to\infty} \left[\sum_{n=0}^{N} x^n - \sum_{n=0}^{N} x^{n+1} \right] = \lim_{N\to\infty} \left[e + \underbrace{x - ex}_{=0} - \underbrace{x^{N+1}}_{\|\cdot\| \le \|x\|^{N+1} \to 0} \right]$ $\Rightarrow x^{N+1} \xrightarrow{N\to\infty} 0$

$= e$, and similarly $(e-x)y = e$.
$\Rightarrow e - x \in G(A)$ and $y = (e-x)^{-1}$. This proves "a)".
But then

$$\|y - e - x\| \overset{\|\cdot\| \text{ contin.}}{=} \lim_{N\to\infty} \left\| \sum_{n=0}^{N} x^n - e - x \right\| = \lim_{N\to\infty} \left\| \sum_{n=2}^{N} x^n \right\|$$

$$\le \lim_{N\to\infty} \sum_{n=2}^{N} \|x\|^n = \sum_{n=2}^{\infty} \|x\|^n = \|x\|^2 \sum_{n'=0}^{\infty} \|x\|^{n'} = \frac{\|x\|^2}{1-\|x\|}.$$

Thus "b" holds. Let h be a complex homom. on A.
If $\lambda \in \mathbb{C}$, $|\lambda| \ge 1$, then $\|\lambda^{-1}x\| \le |\lambda|^{-1}\|x\| \le \|x\| < 1$
$\overset{a)}{\Rightarrow} e - \lambda^{-1}x \in G(A) \overset{16.6.}{\Rightarrow} 0 \ne h(e - \lambda^{-1}x) = \underbrace{h(e)}_{=1} - \lambda^{-1}h(x) \Rightarrow h(x) \ne \lambda$.

$\Rightarrow |h(x)| < 1$. $\square$

16.8. Corollary: If h is a complex homomorphism on a Banach algebra A, then $h \in A^*$ and $|h(x)| \le \|x\|$ $\forall x \in A$.

Proof. If $x = 0$, by linearity of h $\Rightarrow h(x) = 0$. $\Rightarrow |h(x)| = 0 = \|0\|$.
Assume $x \ne 0$. For $0 < \varepsilon < 1$, consider $y := \frac{1-\varepsilon}{\|x\|} x$
$\Rightarrow \|y\| = 1 - \varepsilon < 1 \overset{16.7.c)}{\Rightarrow} 1 > |h(y)| = \frac{1-\varepsilon}{\|x\|} |h(x)|$
$\Rightarrow |h(x)| < \frac{\|x\|}{1-\varepsilon}$ $\forall 0 < \varepsilon < 1$ $\Rightarrow |h(x)| \le \|x\|$. Thus h is linear and bounded $\overset{5.3.}{\Rightarrow} h$ is continuous. $\therefore h \in A^*$ and $|h(x)| \le \|x\|$ $\forall x \in A$. $\square$

16.9. Definition: Consider a Banach algebra $\mathcal{A}$, and $x \in \mathcal{A}$.

* the spectrum of x is $\sigma(x) := \{\lambda \in \mathbb{C} \mid \lambda e - x \notin G(\mathcal{A})\}$
* The complement of spectrum is called the resolvent set $\rho(x) := \{\lambda \in \mathbb{C} \mid \lambda e - x \in G(\mathcal{A})\}$.

  Thus for any $\lambda \in \rho(x)$, the resovent $R_\lambda(x) := (\lambda e - x)^{-1} \in \mathcal{A}$.
* The spectral radius of x is $r_\sigma(x) := \sup_{\lambda \in \sigma(x)} |\lambda|$.

16.10. Lemma: Suppose $\mathcal{A}$ is a Banach algebra and $x \in G(\mathcal{A})$. Then $\|x\|, \|x^{-1}\| > 0$ and for all $y \in \mathcal{A}$ with $\|y\| < \frac{1}{2\|x^{-1}\|}$ we have $x + y \in G(\mathcal{A})$ and

$$(*) \qquad \|(x+y)^{-1} - x^{-1} + x^{-1} y x^{-1}\| \leq 2 \|y\|^2 \|x^{-1}\|^3.$$

Proof. By 16.4., $x \in G(\mathcal{A}) \Rightarrow x \neq 0 \Rightarrow \|x\| > 0$. But then $x^{-1} \in G(\mathcal{A}) \Rightarrow \|x^{-1}\| > 0$. Now $x + y = x(e + x^{-1} y)$.
Since $\|-x^{-1}y\| \leq \|x^{-1}\| \|y\| < \frac{1}{2} \overset{16.7.}{\Longrightarrow} e + x^{-1}y \in G(\mathcal{A})$
and $\|(e + x^{-1}y)^{-1} - e + x^{-1}y\| \leq \frac{\|-x^{-1}y\|^2}{1 - \|-x^{-1}y\|} \leq \|x^{-1}\|^2 \|y\|^2 \cdot \overbrace{\frac{1}{1-\frac{1}{2}}}^{=2}$

Since then $x + y \in G(\mathcal{A})$ and $(x+y)^{-1} = (e + x^{-1}y)^{-1} x^{-1}$
$= [(e + x^{-1}y)^{-1} - e + x^{-1}y] x^{-1} + e x^{-1} - x^{-1} y x^{-1}$
$\Rightarrow \|(x+y)^{-1} - x^{-1} + x^{-1} y x^{-1}\| \leq 2 \|x^{-1}\|^3 \|y\|^2$ $\square$

16.11. Theorem: If $\mathcal{A}$ is a Banach algebra, then $G(\mathcal{A})$ is an open subset of $\mathcal{A}$ and the map $x \mapsto x^{-1}$ is a homeomorphism $G(\mathcal{A})$ onto $G(\mathcal{A})$.

Proof. Suppose $x \in G(\mathcal{A})$. By 16.10. then $B(x, \varepsilon) \subset G(\mathcal{A})$ for $\varepsilon = \frac{1}{2\|x^{-1}\|} > 0$. Thus $G(\mathcal{A})$ is open in $\mathcal{A}$. The map $\Phi(x) = x^{-1}$ is obviously a map $G(\mathcal{A}) \to G(\mathcal{A})$ (see p. 125). Since $(x^{-1})^{-1} = x$, $\Phi^2 = id_{G(\mathcal{A})}$, and thus Φ is bijective, $\Phi^{-1} = \Phi$.
On the other hand, by 16.10. for $x' \in B(x, \varepsilon_0)$, $x \in G(\mathcal{A})$, $\varepsilon_0 = \frac{1}{2\|x^{-1}\|} > 0$, we have $\|\Phi(x') - \Phi(x)\| \leq \|x^{-1}(x' - x)x^{-1}\| + 2\|x' - x\|^2 \|x^{-1}\|^3$
$\leq \|x' - x\| \cdot \|x^{-1}\|^2 (1 + 2\|x' - x\| \|x^{-1}\|) \leq 2 \|x^{-1}\|^2 \|x' - x\|$. Thus $\forall \varepsilon > 0$

and $x' \in B(x, \delta)$, $\delta := \frac{\varepsilon}{4\|x^{-1}\|^2} > 0$, we have $\|\Phi(x') - \Phi(x)\| < \varepsilon$.

Therefore, Φ is continuous $\Rightarrow$ $\Phi^{-1} = \Phi$ is contin.
$\therefore$ Φ is a homeomorphism. $\square$