15.3. Corollaries: Suppose μ is a bounded, $\mathbb{K}$-valued Borel measure on a compact Hausdorff set Q.

a) If $\mathcal{F}$ is a Fréchet space and $f: Q \to \mathcal{F}$ is continuous, then $\exists!\, u \in \mathcal{F}$ s.t.

$$u = \int_Q d\mu\, f, \quad \text{as a VVI.}$$

b) If $\mathcal{B}$ is a Banach space and $f: Q \to \mathcal{B}$ is continuous, then $\|f\|$ is Borel measurable, $\exists!$ VVI $\int_Q d\mu f \in \mathcal{B}$, and

$$(*) \qquad \left\| \int_Q d\mu\, f \right\| \le \int_Q d|\mu|\, \|f\| < \infty.$$

c) If $\mathcal{V}_1, \mathcal{V}_2$ are topol. vect. spaces whose duals separate points on them, $T: \mathcal{V}_1 \to \mathcal{V}_2$ is continuous and linear, $f: Q \to \mathcal{V}_1$ is continuous, and $\overline{\text{Hull}(f(Q))}$ is compact, then $\exists!$ VVIs $\int_Q d\mu\, f \in \mathcal{V}_1$, $\int_Q d\mu\, T \circ f \in \mathcal{V}_2$ and

$$T\left(\int_Q d\mu\, f\right) = \int_Q d\mu\, T \circ f$$

d) If $\mathcal{F}_1, \mathcal{F}_2$ are Fréchet spaces, $T: \mathcal{F}_1 \to \mathcal{F}_2$ is contin. and linear, $f: Q \to \mathcal{F}_1$ is continuous, then

$$T\Big(\underbrace{\int_Q d\mu\, f}_{\in \mathcal{F}_1}\Big) = \underbrace{\int_Q d\mu\, T \circ f}_{\in \mathcal{F}_2}$$

Proof 'a)' $\mathcal{F}$ = Fréchet $\Rightarrow$ locally convex $\Rightarrow$ $\mathcal{F}^*$ separates points on $\mathcal{F}$. (12.7.a) Also f contin. $\Rightarrow$ $f(Q)$ compact in $\mathcal{F}$ $\Rightarrow$ $\overline{\text{Hull}(f(Q))}$ is compact by 14.12. Thus can apply 15.2. $\Rightarrow \exists!\, u \in \mathcal{F}$. ■

b) $\mathcal{B}$ = Banach $\Rightarrow$ Fréchet, and thus by a), $\exists!\, u_0 \in \mathcal{B}$ s.t. $u_0 = \int_Q d\mu\, f$. By 12.4. b) $\exists \Lambda \in \mathcal{B}^*$ s.t. $\|u_0\| = \Lambda u_0$ and $|\Lambda u| \le \|u\| \;\forall u \in \mathcal{B}$ [RCA, 6.12.]

But then

$$\|u_0\| = \Lambda u_0 \overset{\text{VVI}}{=} \int_Q \mu(dx)\, \Lambda(f(x)) \le \int_Q |\mu|(dx)\, |\Lambda(\underbrace{f(x)}_{\in \mathcal{B}})|$$
$$\le \int_Q d|\mu|\, \|f\|, \text{ where } \|f\|: Q \to \mathbb{R} \text{ is continuous}$$

$\Rightarrow$ $\|f\|$ is Borel measurable and $\sup_{x\in Q} \|f(x)\| < \infty$.
Since $|\mu|$ is a positive bounded Borel measure [Rudin, RCA, Thrm 6.4.], this proves $(*)$ ◪

c) By 15.2. $\exists!\ w_0 := \int_Q d\mu\, f \in \mathcal{V}_1$. Also $f_2 := T\circ f : Q \to \mathcal{V}_2$ is continuous, and as in "a)" on p. 114, this implies that f_2 is weakly integrable. If $\Lambda \in \mathcal{V}_2^*$
$\Rightarrow$ $\Lambda\circ T$ is linear, contin. map $\mathcal{V}_1 \to \mathbb{K}$
$\Rightarrow$ $\Lambda\circ T \in \mathcal{V}_1^*$. Thus $(\Lambda\circ T)(w_0) = \Lambda(T(w_0))$
$= \int_Q d\mu\ \Lambda\circ T\circ f = \int_Q d\mu\ \Lambda\circ f_2 \quad \forall \Lambda \in \mathcal{V}_2^*$

$\Rightarrow$ $T(w_0) = \int_Q d\mu\, f_2$ as a V.V.I.,
and $T(w_0) \in \mathcal{V}_2$ ◪

d) Follows from c) and proof of a) □

15.4. Definition (Fréchet derivative)

Suppose $\mathcal{B}_1, \mathcal{B}_2$ are Banach spaces, $\Omega \subset \mathcal{B}_1$ is open and $f: \Omega \to \mathcal{B}_2$. If $a \in \Omega$ is such that there is $T_a \in \mathcal{B}(\mathcal{B}_1, \mathcal{B}_2)$ (as defined in Ex. 8.7.) for which

$$\lim_{x\to 0} \frac{\|f(a+x) - f(a) - T_a x\|}{\|x\|} = 0,$$

then f is (Fréchet)-differentiable at a and $T_a := (Df)_a$ is the Fréchet-derivative of f at a.

* If such T_a exists, it is unique, and f is continuous at a.

15.2.' Example: Let S_1 denote the collection of Schwartz functions $\varphi: \mathbb{R} \to \mathbb{K}$. Then a) S_1 contains smooth functions with compact support, b) S_1 is a vector space, c) $S_1 \subset L^1(\mathbb{R})$, and d) Fourier transform $\varphi \mapsto \hat{\varphi}$, $\hat{\varphi}(k) := \int dx\, e^{i2\pi kx} \varphi(x)$ is a bijection $S_1 \to S_1$. [Proof: Rudin, FA, 7.4 and 7.6, or lecture notes of 2008 course of Introd. to Math. phys, sec. 6.2 & 6.3.]

15.2' cont... By c), can define $\Lambda_\varphi(f) := \int dx\, \check{\varphi}(x) f(x) \in \mathbb{K}$ $\forall \varphi \in S_1$, $f \in \mathcal{V} := L^\infty(\mathbb{R})$. $\Rightarrow \Lambda_\varphi : \mathcal{V} \to \mathbb{K}$ is linear.

By a), $\mathcal{K} := \{\Lambda_\varphi : \mathcal{V} \to \mathbb{K} \mid \varphi \in S_1\}$ separates points on $\mathcal{V}$. (See e.g. Rudin, FA, Sec. 6.1.) Let $T_\mathcal{V}$ = $\mathcal{K}$-weak topology on $\mathcal{V}$. By 13.1. then $\mathcal{V}$ is a locally convex topol. vect. space and $\mathcal{V}^* = \mathcal{K}$ separates points on it. Let μ = Lebesgue measure on $\mathbb{R}$, and define $f : \mathbb{R} \to \mathcal{V}$ by $f_k(x) := e^{-i2\pi k\cdot x}$ $\forall k, x \in \mathbb{R}$. ($\Rightarrow |f_k(x)| = 1 \Rightarrow f_k \in \mathcal{V}\ \forall k$)

Now $\Lambda \in \mathcal{V}^* \Rightarrow \exists \varphi \in S_1$ s.t. $\Lambda(f_k) = \int dx\, \check{\varphi}(x) e^{-i2\pi kx} = \hat{\check{\varphi}}(k)\ \forall k$

$\Rightarrow k \mapsto \Lambda(f_k) \in S_1 \subset L^1(\mu)$. Thus f_k is weakly integrable.

However, there is no $u \in L^\infty(\mathbb{R})$ s.t. $\Lambda(u) = \int dx\, \check{\varphi}(x) u(x)$ $= \int dk\, \Lambda(f_k) = \int dk\, \hat{\check{\varphi}}(k) = \varphi(0)$ $\forall \varphi \in S_1$.

$\therefore \nexists$ WI $\int dk\, f_k$ in $\mathcal{V}$.

(In fact, the map f is continuous, so only the compactness and boundedness assumptions in Thrm. 15.2. fail here.)

15.5. Reminder (complex analysis)

* $\Omega \subset \mathbb{C}$ is a region, if it is $\neq \emptyset$, open, and connected.

* If $\Omega \subset \mathbb{C}$ is open, it is a union of disjoint regions.
* If $\Omega \subset \mathbb{C}$ is open, $z_0 \in \Omega$, and $f : \Omega \to \mathbb{C}$ has a limit $\lim\limits_{\substack{z \to z_0 \\ z \in \Omega \setminus \{z_0\}}} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0)$ for all $z_0 \in \Omega$

  then f is called holomorphic (or analytic) in Ω.
* If f is holomorphic in $\mathbb{C}$, it is called entire.
* A C^1-curve in $\mathbb{C}$ is a map $\gamma : [t_1, t_2] \to \mathbb{C}$, $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$, such that γ is continuous and continuously differentiable on (t_1, t_2).

* A path in $\mathbb{C}$, is a continuous map $\gamma : [t_1, t_2] \to \mathbb{C}$ which can be expressed as a union of finitely many C^1-curves. (= piecewise-C^1 curve)
* $R(\gamma) := \gamma([t_1, t_2]) \subset \mathbb{C}$ is the contour of the path.
* A closed path is a path for which $\gamma(t_1) = \gamma(t_2)$.

* If γ is a path, the following shorthand notations are used: (they make sence, for instance, for every $f : \Omega \to \mathbb{C}$ which is continuous on $R(\gamma) \subset \Omega$)

$$\oint_\gamma dz\, f(z) := \sum_{i=1}^{n} \int_{u_i}^{u_{i+1}} dt\, \gamma_i'(t)\, f(\gamma_i(t)) = \int_{t_1}^{t_2} dt\, \gamma'(t)\, f(\gamma(t))$$

and $$\oint_\gamma |dz|\, f(z) := \sum_{i=1}^{n} \int_{u_i}^{u_{i+1}} dt\, |\gamma_i'(t)|\, f(\gamma_i(t)) = \int_{t_1}^{t_2} dt\, |\gamma'(t)|\, f(\gamma(t)),$$

where u_i, $i = 1, \dots, n+1$, are points such that $t_1 = u_1 < u_2 < \dots < u_{n+1} = t_2$ and $\gamma_i := \gamma|_{[u_i, u_{i+1}]}$ is a C^1-curve.

* If γ is a closed path, $\Omega := \mathbb{C} \setminus R(\gamma)$, and for $z \in \Omega$ set

$$\mathrm{Ind}_\gamma(z) := \frac{1}{2\pi i} \oint_\gamma d\zeta\, \frac{1}{\zeta - z} \quad (= \text{index of } z \text{ w.r.t. } \gamma),$$

then $\mathrm{Ind}_\gamma(z) \in \mathbb{Z}$, $\mathrm{Ind}_\gamma|_{\Omega'} =$ constant for every connected component Ω' of Ω, and $\mathrm{Ind}_\gamma(z) = 0$ if $z \in \Omega_\infty =$ the unbounded component of Ω.
(Proof: Rudin, RCA, 10.10.)

* $\mathrm{Ind}_\gamma(z)$ is also called the "<u>winding number</u>" of z relative to γ.

* If $\gamma(\varphi) := z_0 + \varepsilon e^{i\varphi}$, $\varphi \in [0, 2\pi]$, $\varepsilon > 0$, $z_0 \in \mathbb{C}$
($\Rightarrow$ γ is the positively oriented circle of radius ε at z_0)
then
$$\mathrm{Ind}_\gamma(z) = \begin{cases} 1, & \text{if } |z - z_0| < \varepsilon \\ 0, & \text{if } |z - z_0| > \varepsilon \end{cases} \quad \text{and } R(\gamma) = \{|z - z_0| = \varepsilon\}.$$

* Properties: (References are to theorems in Rudin, RCA.)

a) Every bounded entire function is constant.
(Liouville's theorem, RCA 10.23.)

b) If $\Omega \subset \mathbb{C}$ is <u>open</u>, $f : \Omega \to \mathbb{C}$ is <u>continuous</u>, and

$\oint_{\partial\Delta} dz\, f(z) = 0$ for every closed positively oriented triangle $\underbrace{\partial\Delta}_{\Delta \subset \Omega}$, then f is holom. in Ω.
(Morera's theorem, 10.17.)

c) If $z_0 \in \Omega \subset \mathbb{C}$, Ω open, and f is holomorphic in $\Omega \setminus \{z_0\}$, then one of the following holds

i) f has a removable singularity at z_0
$\Leftrightarrow$ can define $f(z_0)$ such that f is then holom. in Ω

ii) f has a pole of order $m \in \mathbb{N}_+$ at z_0
$\Leftrightarrow$

$$f(z) - \sum_{k=1}^{m} \frac{c_k}{(z-z_0)^k} \text{ has a removable singularity at } z_0$$

(for some $c_k \in \mathbb{C}$, $k = 1, \dots, m$)

iii) f has an essential singularity at z_0
$\Leftrightarrow$

$\forall \varepsilon > 0$ s.t. $B(z_0, \varepsilon) \subset \Omega$ the set $f(B(z_0, \varepsilon) \setminus \{z_0\})$ is dense in $\mathbb{C}$.

(Proof : 10.21)

d) If $\Omega \subset \mathbb{C}$ is a region and f is holomorphic in Ω, then $|f|$ has no local maximum in Ω, unless f = constant. (10.24.)

e) Cauchy's theorem holds : (10.35)

Suppose $\Omega \subset \mathbb{C}$ is open and f is holomorphic in Ω. If γ is a closed path with $R(\gamma) \subset \Omega$ and $\mathrm{Ind}_\gamma(z) = 0$ for $z \notin \Omega$, then

$$\oint_\gamma d\zeta\, f(\zeta) = 0 \qquad \text{and}$$

$$\frac{1}{2\pi i} \oint_\gamma d\zeta \frac{1}{\zeta - z_0} f(\zeta) = \mathrm{Ind}_\gamma(z_0) \cdot f(z_0) \quad \forall z_0 \in \Omega \setminus R(\gamma)$$

In addition, if γ_1 and γ_2 are closed paths with $R(\gamma_1)$, $R(\gamma_2) \subset \Omega$, and

$$(*) \qquad \mathrm{Ind}_{\gamma_1}(z) = \mathrm{Ind}_{\gamma_2}(z) \quad \forall z \notin \Omega$$

then $\oint_{\gamma_1} d\zeta\, f(\zeta) = \oint_{\gamma_2} d\zeta\, f(\zeta)$.

f) If (Ω is a region and) γ_1 and γ_2 are homotopic in Ω, then $(*)$ holds. (10.40)

15.6 Definition: Assume $\Omega \subset \mathbb{C}$ is open, and V is a complex topol. vect. space.
Consider some $f : \Omega \to V$.

i) If $\Lambda \circ f$ is holomorphic in Ω $\forall \Lambda \in V^*$, then f is weakly holomorphic in Ω.

ii) If $\lim\limits_{\substack{z \to z_0 \\ z \in \Omega \setminus \{z_0\}}} \frac{1}{z - z_0}(f(z) - f(z_0))$ exists (in the topology of V) for all $z_0 \in \Omega$, then f is strongly holomorphic in Ω. (the limits define derivative $f' : \Omega \to V$)

15.7 → * Obviously, every strongly holomorphic function is weakly holomorphic: $\Lambda \in V^* \Rightarrow \Lambda$ contin. and linear $\Rightarrow$ $\exists \lim\limits_{z \to z_0} \frac{1}{z - z_0}(\Lambda(f(z)) - \Lambda(f(z_0))) = \Lambda(f'(z_0))$. The converse holds in Fréchet spaces:

15.8. Theorem: Suppose $\Omega \subset \mathbb{C}$ is open, V is a complex Fréchet space, and that $f : \Omega \to V$ is weakly holomorphic. Then all of the following hold:

a) f is (strongly) continuous on Ω
b) All statements of the Cauchy theorem in "e)" hold when the integrals are defined as VVIs.
c) f is strongly holomorphic on Ω, and for $z_0 \in \Omega$, $f'(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{d\zeta}{(\zeta - z_0)^2} f(\zeta)$
where γ is any path which is homotopic to $\circlearrowleft_{z_0}$ in $\Omega_{c(z_0)} \setminus \{z_0\}$, $\Omega_{c(z_0)}$ = conn. compo.

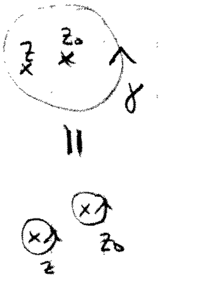
Proof: "a)" Suppose $z_0 \in \Omega$. It suffices to prove f is contin. at z_0.
$\Omega \subset \mathbb{C}$ open $\Rightarrow \exists \varepsilon > 0$ s.t. $\overline{B(z_0, 2\varepsilon)} \subset \Omega$. Let γ be the positively oriented circle of radius 2ε at z_0. Fix $\Lambda \in V^*$. By assumption, $\Lambda \circ f$ is holom. in Ω $\Rightarrow$ also $g(z) := \frac{\Lambda(f(z)) - \Lambda(f(z_0))}{z - z_0}$ is holom. in Ω (power series at z_0)

$$\overset{\text{Cauchy}}{\Rightarrow} \quad g(z) = \frac{1}{2\pi i} \oint_\gamma d\zeta \, \frac{g(\zeta)}{\zeta - z} \quad \forall\, |z - z_0| < 2\varepsilon$$

$$\Rightarrow \quad \frac{\Lambda(f(z)) - \Lambda(f(z_0))}{z - z_0} = \frac{1}{2\pi i} \oint_\gamma d\zeta \, \frac{\Lambda(f(\zeta))}{(\zeta - z)(\zeta - z_0)} \quad \forall\, 0 < |z - z_0| < 2\varepsilon$$

Residue theorem, $z \neq z_0$

since $\frac{1}{2\pi i}\oint_\gamma d\zeta \frac{1}{(\zeta-z)(\zeta-z_0)} \overset{\text{b}}{=} \frac{1}{z-z_0}+\frac{1}{z_0-z}=0.$



Since $\Lambda\circ f$ is contin. on Ω an $R(\gamma)$ = compact $\Rightarrow \exists\, M(\Lambda) := \max_{z\in R(\gamma)} |\Lambda(f(z))|$. If $0<|z-z_0|\leq\varepsilon$

and $\zeta\in R(\gamma) \Rightarrow |\zeta-z| \geq |\zeta-z_0|-|z-z_0| \geq 2\varepsilon-\varepsilon=\varepsilon$, and thus

$$\left|\Lambda\left(\frac{f(z)-f(z_0)}{z-z_0}\right)\right| \leq \frac{1}{2\pi}\oint_\gamma |d\zeta| \frac{M(\Lambda)}{\varepsilon|\zeta-z_0|} = \frac{M(\Lambda)}{\varepsilon} < \infty$$

By 13.5.c) $\Rightarrow$ $E := \left\{ \frac{1}{z-z_0}(f(z)-f(z_0)) \,\middle|\, 0<|z-z_0|\leq\varepsilon \right\}$

is weakly bounded $\overset{13.14.}{\Longrightarrow}$ E is originally bounded (V is loc. convex!)

Thus if $0\in V_0 \in \mathcal{T}_V$ balanced $\Rightarrow \exists\, t>0$ s.t. $E\subset tV_0$

$\Rightarrow f(z)-f(z_0) \in t(z-z_0)V_0 \quad \forall\, 0<|z-z_0|\leq\varepsilon$

$\Rightarrow \quad \forall\, |z-z_0|<\varepsilon_0 := \min(\varepsilon, \frac{1}{t})$ we have $t|z-z_0|\leq 1$ and thus $f(z)\in f(z_0)+V_0$. By 4.12. $\Rightarrow$ f contin. at z_0.

$\therefore$ f is continuous on Ω. ☐

"b)" Since γ is contin., by "a)" also $f\circ\gamma: [t_1,t_2]\to V$ is continuous. Thus by 15.3.a) all the integrals in the Cauchy theorem on p. 120 are well-defined VVIs.(*) Choosing an arbitrary $\Lambda\in V^*$ and using the standard Cauchy to the holomorphic function $\Lambda\circ f$ proves then that the identities also hold between the VVIs.

For instance, $\oint_\gamma d\zeta\, \Lambda(f(\zeta)) = 0 = \Lambda 0 \quad \forall \Lambda\in V^* \Rightarrow \oint_\gamma d\zeta f(\zeta)=0.$

(*) Note: If $\alpha: Q\to\mathbb{k}$, $g: Q\to V$ are contin. $\Rightarrow (\alpha,g): Q\to\mathbb{k}\times V$ contin. $\overset{\text{V.T.V.S.}}{\Longrightarrow} \alpha g: Q\to V$ contin.

"c)" Continuing from "a)": Define $v_0 := \frac{1}{2\pi i}\oint_\gamma d\zeta \frac{1}{(\zeta-z_0)^2} f(\zeta) \in V$

as a VVI. (note that on $R(\gamma)$, $|\zeta-z_0|=2\varepsilon>0$.) Then by "b)"

$\forall\, 0<|z-z_0|<2\varepsilon$:

$$f(z)-f(z_0) = \frac{1}{2\pi i}\oint_\gamma d\zeta \left[\frac{1}{\zeta-z}-\frac{1}{\zeta-z_0}\right] f(\zeta)$$

where $[\,] = \frac{\zeta-z_0-(\zeta-z)}{(\zeta-z)(\zeta-z_0)} = (z-z_0)\frac{1}{(\zeta-z_0)^2}\left(\underbrace{\frac{\zeta-z_0}{\zeta-z}-1}_{=\frac{z-z_0}{\zeta-z}}+1\right).$

thus for $0<|z-z_0|<2\varepsilon$,

$$\frac{1}{z-z_0}(f(z)-f(z_0)) = v_0 + (z-z_0)g(z), \text{ where, explicitly,}$$

$$g(z) := \underbrace{\frac{1}{2\pi}\int_0^{2\pi} d\varphi}_{\text{probability measure on } [0,2\pi]} \frac{1}{2\varepsilon e^{i\varphi}} \frac{1}{2\varepsilon e^{i\varphi}+z_0-z} f(z_0+2\varepsilon e^{i\varphi}).$$

Suppose $0 \in V_0 \in \mathcal{T}_N$, V_0 is convex and balanced. Set $K = f(R(\gamma)) = \text{compact} \overset{4.14.b)}{\Longrightarrow} \exists t > 0$ s.t. $K \subset t V_0$. Then if $0 < |z - z_0| \le \varepsilon \Rightarrow \left| \frac{1}{2\varepsilon e^{i\varphi}} \frac{1}{2\varepsilon e^{i\varphi} + z_0 - z} \right| \le \frac{1}{2\varepsilon} \frac{1}{\varepsilon} = \frac{1}{2} \frac{1}{\varepsilon^2}$

and thus $g(z) = \int_0^{2\pi} \frac{d\varphi}{2\pi} F(\varphi)$ where $F(\varphi) \in t\varepsilon^{-2} V_0 \ \forall \varphi$.

Since F is contin. can apply 15.2. $\Rightarrow g(z) \in \overline{\text{Hull}(F([0,2\pi]))}$
$\subset \overline{\text{Hull}(t\varepsilon^{-2} V_0)} = \overline{t\varepsilon^{-2}\text{Hull}(V_0)} = t\varepsilon^{-2}\overline{V_0}$, as V_0 is convex and $v \mapsto t\varepsilon^{-2} v$ is homeomorphism. Thus $\forall\, 0 < |z - z_0| < \min(\frac{\varepsilon^2}{t}, \varepsilon)$
$\frac{1}{z - z_0}(f(z) - f(z_0)) \in v_0 + (z - z_0) t\varepsilon^{-2} \overline{V_0} \underset{\overline{V_0} \text{ balanced}}{\subset} v_0 + \overline{V_0}$.

By 4.9. and 4.12. b) this implies that $\frac{1}{z - z_0}(f(z) - f(z_0)) \to v_0$ in V. Thus f is strongly holom. in Ω and $f'(z_0) = v_0$. For $z_0 \in \Omega$, let $\Omega_{C(z_0)} :=$ connected component of Ω containing z_0 and $\Omega' := \Omega_{C(z_0)} \setminus \{z_0\}$. Thus $\Omega' =$ region and $\zeta \mapsto \frac{1}{(\zeta - z_0)^2} f(\zeta)$ is weakly holom. in Ω' $\overset{\text{"b)"}}{\Longrightarrow}$ γ allows for Ω'-homotopic modifications. □

<u>15.7. Proposition</u>: Suppose N is a complex topol. vect. space on which N^* separates points. If $f: \mathbb{C} \to N$ is weakly holomorphic and $f(\mathbb{C}) \subset N$ is weakly bounded, then f is constant.

<u>Proof.</u> If $\Lambda \in N^* \Rightarrow \Lambda \circ f$ is entire and by 13.5. c)
$\sup_{z \in \mathbb{C}} |\Lambda(f(z))| < \infty \underset{\text{Liouville}}{\Longrightarrow} \Lambda \circ f = \text{constant}$.

$\therefore \Lambda(f(z)) = \Lambda(f(0)) \ \forall z \in \mathbb{C}, \Lambda \in N^*$
$\Rightarrow f(z) = f(0) \ \forall z \in \mathbb{C}$ since N^* separates points. □

Appendix:

<u>15.2".</u> Note that, if f_1, f_2 are weakly integrable and the VVI's $\int d\mu\, f_1$ and $\int d\mu\, f_2$ exist, then also $\alpha f_1 + \beta f_2$ is weakly integrable and

$$\int_Q d\mu\, (\alpha f_1 + \beta f_2) = \alpha \int_Q d\mu\, f_1 + \beta \int_Q d\mu\, f_2.$$

(Proof: If $\Lambda \in N^* \Rightarrow \Lambda \circ f_1, \Lambda \circ f_2 \in L^1(|\mu|) \Rightarrow \alpha \Lambda \circ f_1 + \beta \Lambda \circ f_2 \in L^1(|\mu|)$.
But $(\Lambda \circ (\alpha f_1 + \beta f_2))(x) = \Lambda(\alpha f_1(x) + \beta f_2(x)) = (\alpha \Lambda \circ f_1 + \beta \Lambda \circ f_2)(x)$
$\Rightarrow \int_Q d\mu\, \Lambda \circ (\alpha f_1 + \beta f_2) = \alpha \Lambda(\int d\mu\, f_1) + \beta \Lambda(\int d\mu\, f_2)$ □)