* For further use, let use use the notation Δ_d, $d \in \mathbb{N}_+$, for the standard $(d-1)$-dimensional simplex :

$$\Delta_d := \{ t \in [0,1]^d \mid \sum_{i=1}^{d} t_i = 1 \}.$$

14.7. Lemma: $\Delta_d \subset \mathbb{R}^d$ is compact.

Proof: The map $f: \mathbb{R}^d \to \mathbb{R}$ def. by $f(t) = \sum_{i=1}^{d} t_i$ is continuous ($\in (\mathbb{R}^d)^*$).
$\Rightarrow$ $\Delta_d = [0,1]^d \cap f^{\leftarrow}\{1\}$ is closed in $\mathbb{R}^d$. It is obviously bounded $\Rightarrow$ Δ_d is compact. $\square$

14.8. Proposition: Suppose $d \in \mathbb{N}_+$ and $K \subset \mathbb{R}^d$ is compact. Set $H := \mathrm{Hull}(K)$. Then $\overline{H} = H$ and H is compact.

Proof: By Tychonoff's theorem and 14.7. the set $\Sigma := \Delta_{d+1} \times K^{d+1} \subset \mathbb{R}^{d+1} \times (\mathbb{R}^d)^{d+1} = \mathbb{R}^{(d+1)^2}$ is compact. The map $\Phi: \mathbb{R}^{(d+1)^2} \to \mathbb{R}^d$ def. by $\Phi(t, x_1, \dots, x_{d+1}) := \sum_{i=1}^{d+1} t_i x_i \in \mathbb{R}^d$ is obviously contin.

Thus $R := \Phi(\Sigma)$ is compact. Every $x \in R$ is a convex combination of elements of K $\Rightarrow$ $R \subset H$. If $x \in H$, the following Lemma $\Rightarrow \exists N \leq d+1$, $u \in \Delta_N$, $y_i \in K$, $i = 1, \dots, N$, s.t. $x = \sum_{i=1}^{N} u_i y_i$
Set $y_i = y_N$, $t_i = 0$ for $N < i \leq d+1$, and $t_i = u_i$ for $i \leq N$.
Then $\sum_{i=1}^{d+1} t_i = \sum_{i=1}^{N} u_i = 1$ and thus $t \in \Delta_{d+1}$, $y \in K^{d+1}$

and $x = \sum_{i=1}^{d+1} t_i y_i \in R$. Thus $H \subset R$. $\therefore H = R$ = compact $\Rightarrow \overline{H} = H$ $\square$

14.9. Lemma: Suppose $d \in \mathbb{N}_+$, $E \subset \mathbb{R}^d$ and $y_0 \in \mathrm{Hull}(E)$. Then $\exists N \in \mathbb{N}_+$, $N \leq d+1$, and $x \in E^N$, $t \in \Delta_N$ s.t. $y_0 = \sum_{i=1}^{N} t_i x_i$.

Proof. $y_0 \in \mathrm{Hull}(E) \Rightarrow \exists n \in \mathbb{N}_+$, $x \in E^n$, $t \in \Delta_n$ s.t. $y_0 = \sum_{i=1}^{n} t_i x_i$
If $n \leq d+1$, we can choose $N = n$. Suppose that $n > d+1$. We claim that then $\exists i_0 \in \{1, \dots, n\}$ and $t' \in \Delta_n$

s.t. $t'_{i_0} = 0$ and $\sum_{i=1}^n t_i x_i = \sum_{i=1}^n t'_i x_i$.

$\Rightarrow \sum_{i=1}^n t_i x_i = \sum_{i=1}^{n-1} u_i x'_i$ where $x'_i = \begin{cases} x_i, & i < i_0 \\ x_{i+1}, & i \geq i_0 \end{cases}$

and $u_i = \begin{cases} t'_i, & i < i_0 \\ t'_{i+1}, & i \geq i_0 \end{cases}$. As $\sum_{i=1}^{n-1} u_i = \sum_{i=1}^{n} t'_i = 1$

$\Rightarrow u \in \Delta_{n-1}$, $x' \in K^{n-1}$ and $y_0 = \sum_{i=1}^{n-1} u_i x'_i$. Thus

this procedure can be iterated until $n-1 = d+1$. But then the result holds with $N = d+1$, "t" $= u$ and "x" $= x'$.

Thus we only need to prove the iteration step. If $t_{i_0} = 0$ for some i_0 we can choose $t' = t$. Otherwise, $t_i > 0$ $\forall i$. Since $n-1 > d$, the vectors $x_i - x_n$, $i = 1, \ldots, n-1$, are linearly dependent. $\Rightarrow \exists \alpha_i \in \mathbb{R}$, $i = 1, \ldots, n-1$, s.t. $\alpha_i \neq 0$ for some i, and $0 = \sum_{i=1}^{n-1} \alpha_i (x_i - x_n)$. Set $\alpha_n := -\sum_{i=1}^{n-1} \alpha_i$

$\Rightarrow \sum_{i=1}^n \alpha_n = 0$ and $\sum_{i=1}^n \alpha_i x_i = 0$. Choose i_0 s.t.

$\left|\frac{\alpha_i}{t_i}\right| \leq \left|\frac{\alpha_{i_0}}{t_{i_0}}\right|$ $\forall i \leq n$. $\Rightarrow \alpha_{i_0} \neq 0$ (else $\alpha_i = 0$ $\forall i$ ↯)

$\Rightarrow$ can define $t'_i := t_i - \frac{\alpha_i}{\alpha_{i_0}} t_{i_0}$ $\forall i \leq n$.

$\Rightarrow t'_{i_0} = 0$ and $t'_i = t_i \Big(1 - \underbrace{\frac{t_{i_0}}{\alpha_{i_0}} \frac{\alpha_i}{t_i}}_{|\,| \leq 1}\Big) \geq 0$ $\forall i$.

In addition, $\sum_{i=1}^n t'_i = \underbrace{\sum_{i=1}^n t_i}_{=1} - \frac{t_{i_0}}{\alpha_{i_0}} \underbrace{\sum_{i=1}^n \alpha_i}_{=0} = 1 \Rightarrow t'_i \leq 1$ $\forall i$,

and $\sum_{i=1}^n t'_i x_i = \sum_{i=1}^n t_i x_i - \frac{t_{i_0}}{\alpha_{i_0}} \underbrace{\sum_{i=1}^n \alpha_i x_i}_{=0} = \sum_{i=1}^n t_i x_i$. $\square$

<u>14.10. Definition</u>: Consider a topol. vect. space V. A set $E \subset V$ is called <u>totally bounded</u>, if $\forall V_0 \in \mathcal{T}$ with $0 \in V_0$ $\exists F \subset V$ s.t. F is a <u>finite</u> set and $E \subset F + V_0$.

14.11. Proposition: Suppose $\mathcal{V}$ is a topol. vect. space and $E \subset \mathcal{V}$.

a) If E is totally bounded, then E is (topol.) bounded.
b) If E is compact, it is totally bounded.
c) Assume, in addition, that $\mathcal{T}_{\mathcal{V}}$ is induced by an invariant metric d. Then E is totally bounded iff $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}_+$, $x \in \mathcal{V}^N$, s.t. $E \subset \bigcup_{i=1}^{N} B(x_i, \varepsilon)$.

Proof. Exercise 9.1. $\square$

14.12. Theorem Suppose $\mathcal{V}$ is a locally convex topol. vect. space. If $E \subset \mathcal{V}$ is totally bounded, then $\mathrm{Hull}(E)$ is totally bounded.

Proof. Consider $U \in \mathcal{T}_{\mathcal{V}}$ with $0 \in U$. As $\mathcal{V}$ is locally convex $\overset{4.10.}{\Rightarrow}$ $\exists V_0 \in \mathcal{T}_{\mathcal{V}}$ s.t. $0 \in V_0$, V_0 is convex, and $V_0 + V_0 \subset U$.
Now $\exists N \in \mathbb{N}_+$, $y \in \mathcal{V}^N$ s.t. $E \subset F_1 + V_0$ where $F_1 = \{y_i\}_{i=1}^N \subset \mathcal{V}$.
Denote $H_1 := \mathrm{Hull}(F_1)$. If $x \in H_1 \Rightarrow \exists M \in \mathbb{N}_+$, $t \in \Delta_M$, and $i_j \in \{1, \dots, N\}$, $j = 1, \dots, M$, s.t. $x = \sum_{j=1}^{M} t_j y_{i_j}$

$$\Rightarrow x = \sum_{i=1}^{N} y_i \underbrace{\sum_{j=1}^{M} t_j \mathbb{1}(i_j = i)}_{=: \tau(i)} = \sum_{i=1}^{N} \tau(i) y_i.$$ Here $0 \le \tau(i) \le 1$

and $\sum_{i=1}^{N} \tau(i) = \sum_{j=1}^{M} t_j = 1 \Rightarrow \tau \in \Delta_N$. This implies that

$H_1 = \Phi(\Delta_N)$ for the map $\Phi : \mathbb{R}^N \to \mathcal{V}$ def. by $\Phi(t) = \sum_{i=1}^{N} t_i y_i$. Since $\mathcal{V}$ is a topol. vect. space, Φ is contin. As Δ_N is compact $\Rightarrow$ H_1 is compact. $\overset{14.11, b)}{\Rightarrow}$ H_1 is totally bounded. $\Rightarrow$ $\exists F \subset \mathcal{V}$ s.t. $|F| < \infty$ and $H_1 \subset F + V_0$.

Set then $H := \mathrm{Hull}(E)$. If $\tilde{x} \in H \Rightarrow \exists N' \in \mathbb{N}_+$, $t \in \Delta_{N'}$, $x \in E^{N'}$ s.t. $\tilde{x} = \sum_{i=1}^{N'} t_i x_i$. Since $x_i \in E \subset F_1 + V_0 \Rightarrow \forall i$ $\exists l_i \in \{1, \dots, N\}$ s.t. $x_i - y_{l_i} \in V_0$. Thus $\tilde{x} = \sum_{i=1}^{N'} t_i (x_i - y_{l_i}) + \sum_{i=1}^{N'} t_i y_{l_i}$ $\in \mathrm{Hull}(V_0) + \mathrm{Hull}(F_1) \overset{V_0 \text{ convex}}{=} V_0 + H_1$. $\therefore$ $H \subset H_1 + V_0 \subset F + V_0 + V_0 \subset F + U$ $\square$

* The following result generalizes 14.8. to topol. vect. spaces.

14.13. Theorem: Suppose $\mathcal{V}$ is a Fréchet space. If $K \subset \mathcal{V}$ is compact, then $\mathrm{Hull}(K)$ is compact.

<u>Proof</u>: Set $H := \mathrm{Hull}(K)$. Now K = compact $\overset{14.11, b)}{\Rightarrow}$ K totally bounded $\overset{14.12.}{\Rightarrow}$ H totally bounded (since $\mathcal{V}$ = Fréchet $\Rightarrow$ $\mathcal{V}$ loc. convex.) Let d denote the complete invariant metric compatible with the topology of $\mathcal{V}$, and set $B_\varepsilon := B(0,\varepsilon) \subset \mathcal{V}$. Then $\{B_\varepsilon\}_{\varepsilon>0}$ forms a local base for $\mathcal{T}_\mathcal{V}$. Also, by 14.11, c),
$\forall \varepsilon > 0$ $\exists N_\varepsilon \in \mathbb{N}_+$, $x_i(\varepsilon) \in \mathcal{V}$, $i = 1, \dots, N_\varepsilon$, s.t. $H \subset \bigcup_{i=1}^{N_\varepsilon} B_\varepsilon(i)$, where $B_\varepsilon(i) := B_d(x_i(\varepsilon), \varepsilon)$. Thus
$\overline{H} \subset \bigcup_{i=1}^{N_\varepsilon} \overline{B_\varepsilon(i)} \Rightarrow \overline{H} = \bigcup_{i=1}^{N_\varepsilon} K_\varepsilon(i)$ where $K_\varepsilon(i) := \overline{H} \cap \overline{B_\varepsilon(i)}$ is closed in $\mathcal{V}$.

Consider then an open cover Γ of $\overline{H}$. Suppose that no finite subset of Γ covers $\overline{H}$. By the above results there must be some $K_1(i)$ s.t. no finite subset of Γ covers it. Choose one and denote $\tilde{K}_1 := K_1(i)$. As $\tilde{K}_1 \subset \overline{H} \Rightarrow \tilde{K}_1 = \bigcup_{i=1}^{N_\varepsilon} (\tilde{K}_1 \cap K_\varepsilon(i))$ and thus

$\exists \tilde{K}_2 \subset \overline{H}$ such that $\tilde{K}_2 = \tilde{K}_1 \cap K_{\frac{1}{2}}(i)$ and $\tilde{K}_2$ cannot be covered by finite subsets of Γ. Continuing this iteration construction yields sets $\tilde{K}_n \subset \overline{H}$, $n \in \mathbb{N}_+$, for which $\tilde{K}_{n+1} = \tilde{K}_n \cap K_{\frac{1}{n+1}}(i)$ for some $i \in \{1, \dots, N_{\frac{1}{n+1}}\}$ and which cannot be covered by finite subsets of Γ
$\Rightarrow$ each $\tilde{K}_n \neq \emptyset$. For $n \in \mathbb{N}_+$, choose $x_n \in \tilde{K}_n$.

If $n, m \in \mathbb{N}_+$, then $x_n, x_{n+m} \in \tilde{K}_n \Rightarrow$
$d(x_n, x_{n+m}) \leq d(x_n, x_i(\frac{1}{n})) + d(x_i(\frac{1}{n}), x_{n+m}) \leq \frac{2}{n}$.
Thus (x_n) is Cauchy $\Rightarrow \exists x \in \mathcal{V}$ s.t. $x_n \to x$.
If $n \in \mathbb{N}_+$, then $x_{n+m} \in \tilde{K}_n$ $\forall m \geq 0$ $\Rightarrow$ $x \in \overline{\tilde{K}_n} = \tilde{K}_n$.
Therefore, $x \in \bigcap_{n \in \mathbb{N}_+} \tilde{K}_n \subset \overline{H}$. $\Rightarrow \exists V \in \Gamma$ s.t. $x \in V$.

$\Rightarrow \exists \varepsilon > 0$ s.t. $B(x, \varepsilon) \subset V$. Choose $n_0 \in \mathbb{N}_+$ so large that $d(x, x_{n_0}) < \frac{\varepsilon}{3}$ and $n_0 \geq \frac{3}{\varepsilon}$. Then if $y \in \tilde{K}_{n_0}$
$\Rightarrow d(x,y) \leq d(x, x_{n_0}) + d(x_{n_0}, x_i(\frac{1}{n_0})) + d(x_i(\frac{1}{n_0}), y)$
$< \frac{\varepsilon}{3} + \frac{1}{n_0} + \frac{1}{n_0} \leq 3 \cdot \frac{\varepsilon}{3} = \varepsilon \Rightarrow y \in B(x, \varepsilon)$.
Thus $\tilde{K}_{n_0} \subset B(x,\varepsilon) \subset V \in \Gamma$ ↯.

$\therefore$ Γ has a finite subcover. $\therefore$ $\overline{H}$ is compact. □

15. "Vector-valued" integration and holomorphic functions

15.1. Definition: Suppose $\mathcal{V}$ is a topol. vect. space on which $\mathcal{V}^*$ separates points.
Consider a measure μ on a set Q, and assume that $f: Q \to \mathcal{V}$ is such that $\Lambda \circ f \in L^1(|\mu|)$ $\forall \Lambda \in \mathcal{V}^*$.
If $\exists \mathcal{Y} \in \mathcal{V}$ s.t.

$$\Lambda \mathcal{Y} = \int_Q \mu(dx)\, \Lambda(f(x)) \qquad \forall \Lambda \in \mathcal{V}^*,$$

then we denote $\mathcal{Y} = \int_Q d\mu\, f = \int_Q \mu(dx)\, f(x)$.

* These are called "vector valued integrals" in $\mathcal{V}$. The functions f above are called weakly integrable and $\mathcal{Y}$ can be thought of as a "weak integral" of f. Note the similarity to the weak topology defined earlier.
* For $\mathcal{V} = \mathbb{K}^d$, $\mathcal{Y}_i = \int_Q \mu(dx)\, f_i(x)$ $\forall i$, i.e. integrated componentwise.
* Since $\mathcal{V}^*$ is assumed to separate points on $\mathcal{V}$, such $\mathcal{Y}$ are obviously unique. The difficulty is to prove the existence of a weak integral $\mathcal{Y}$. The next theorem gives sufficient conditions for this.

15.2. Theorem Suppose $\mathcal{V}$ is a topol. vect. space on which $\mathcal{V}^*$ separates points.
Consider a bounded Borel $\mathbb{K}$-valued measure μ on a compact Hausdorff space Q.

If $f: Q \to \mathcal{V}$ is continuous and $\mathrm{Hull}(f(Q))$ is compact, then f is weakly integrable and the vector valued integral (VVI) $\int_Q d\mu\, f$ exists.
In addition, if μ is a probability measure, then $\int_Q d\mu\, f \in \overline{\mathrm{Hull}(f(Q))}$.

Proof: a) b) Let us first assume that $\mathbb{K} = \mathbb{R}$ and μ is a Borel probability measure $\Leftrightarrow$ $\int d\mu = 1$ & μ is a positive measure on the Borel σ-algebra of Q
= smallest σ-algebra containing all open sets of Q

a) If $\Lambda \in V^*$ $\Rightarrow$ $\Lambda : V \to \mathbb{K}$ continuous
$\Rightarrow$ $\Lambda \circ f : Q \to \mathbb{K}$ is continuous
$\Rightarrow$ Borel measurable. Also, since then $\Lambda(f(Q))$ is compact in $\mathbb{K}$ $\Rightarrow$ $\exists M < \infty$ s.t. $|\Lambda(f(x))| \leq M$ $\forall x \in Q$. $\Rightarrow$
$\int |\mu|(dx)\, |\Lambda(f(x))| \leq M \int |\mu|(dx) < \infty$ since μ is bounded. Thus $\Lambda \circ f \in L^1(|\mu|)$ $\forall \Lambda \in V^*$.
$\Rightarrow$ f is weakly integrable.

... Denote $H = \mathrm{Hull}(f(Q))$. We need to prove that $\exists v \in \overline{H}$ s.t.

$$(*) \quad \Lambda v = \int_Q \mu(dx)\, \Lambda(f(x)) \qquad \forall \Lambda \in V^*.$$

For any $\Lambda \in V^*$, let $E_\Lambda := \{ v \in \overline{H} \mid \Lambda v = \int d\mu\, \Lambda \circ f \}$.
$\Rightarrow$ $E_\Lambda = \overline{H} \cap (\Lambda^{\leftarrow}\{c_\Lambda\})$ where $c_\Lambda := \int d\mu\, \Lambda \circ f \in \mathbb{R}$ by a).
Since Λ is contin. $\Rightarrow$ $\Lambda^{\leftarrow}\{c_\Lambda\}$ is closed $\Rightarrow$ E_Λ is closed,
& $E_\Lambda \subset \overline{H}$ = compact $\Rightarrow$ E_Λ is compact.
Consider $L \subset V^*$ with $0 < |L| < \infty$. We claim that always $\bigcap_{\Lambda \in L} E_\Lambda \neq \emptyset$. As then each E_Λ is compact and nonempty and the collection $\{E_\Lambda\}_{\Lambda \in V^*}$ has the finite intersection property $\Rightarrow$ $\bigcap_{\Lambda \in V^*} E_\Lambda \neq \emptyset$ (as on p. 107)

$\Rightarrow$ $\exists v \in \bigcap_{\Lambda \in V^*} E_\Lambda$ $\Rightarrow$ $v \in \overline{H}$ and $\Lambda v = \int d\mu\, \Lambda \circ f$ $\forall \Lambda \in V^*$.

Thus it suffices to prove that $\bigcap_{\Lambda \in L} E_\Lambda \neq \emptyset$ for any $L = \{\Lambda_i\}_{i=1}^N$, $\Lambda_i \neq \Lambda_j$, $\Lambda_i \in V^*$, $1 \leq N < \infty$. ($N = |L|$)
Consider the map $\Phi : V \to \mathbb{R}^N$ defined by $(\Phi v)_i := \Lambda_i v \in \mathbb{R}$. Set $K = \Phi(f(Q))$, and define $a \in \mathbb{R}^N$ by $a_i := \int_Q \mu(dx)\, \Lambda_i(f(x))$ $\forall i = 1, \dots, N$.
We claim that $a \in \mathrm{Hull}(K) =: H_K$.
Consider $t \in \mathbb{R}^N \setminus H_K$. As Φ is continuous, $\Phi \circ f$ is continuous $\Rightarrow$ K is compact. Then by 14.8. H_K is a compact convex set. By 12.5.b) $\Rightarrow$ $\exists \tilde{\Lambda} \in (\mathbb{R}^N)^*$ & $\gamma_1, \gamma_2 \in \mathbb{K}$ s.t. $\tilde{\Lambda} y < \gamma_1 < \gamma_2 < \tilde{\Lambda} t$ $\forall y \in H_K$. As noted on p. 95. now $\exists b \in \mathbb{R}^N$ s.t. $\tilde{\Lambda} y = b \cdot y$. Since $K \subset H_K$ $\Rightarrow$

$$\sum_{i=1}^N b_i \Lambda_i(f(x)) < \gamma_1 < \gamma_2 < \sum_{i=1}^N b_i t_i \quad \forall x \in Q.$$

Integrating over the positive measure μ yields

$$b \cdot a = \sum_{i=1}^{N} b_i a_i = \int \mu(dx) \sum_{i=1}^{N} b_i \Lambda_i(f(x)) \leq \gamma_1 \int \mu(dx) = \gamma_1$$

$\Rightarrow b \cdot a < b \cdot t \Rightarrow t \neq a$.

Thus $a \in H_k$. Now if $v \in H = \mathrm{Hull}(f(Q)) \Rightarrow v = \sum_{i=1}^{M} t_i f(x_i)$

Φ linear

$\Rightarrow \Phi(v) = \sum_{i=1}^{M} t_i \underbrace{\Phi(f(x_i))}_{\in K} \in \mathrm{Hull}(K) = H_k$. Conversely,

$y \in H_k \Rightarrow y = \sum_{i=1}^{M} t_i \Phi(f(x_i)) = \Phi\left(\sum_{i=1}^{M} t_i f(x_i)\right) \in \Phi(H)$

Thus $H_k = \Phi(H) \Rightarrow \exists v \in H$ s.t. $a = \Phi(v)$

$\Rightarrow \forall i = 1, \dots, N : \int \mu(dx) \Lambda_i(f(x)) = \Lambda_i v$

$\Rightarrow v \in \bigcap_{\Lambda \in L} E_\Lambda \Rightarrow \bigcap_{\Lambda \in L} E_\Lambda \neq \emptyset$.

$\therefore$ The theorem holds if μ = Borel prob. measure and $\mathbb{K} = \mathbb{R}$.

c) If $\mathbb{K} = \mathbb{R}$ and μ = real bounded Borel measure $\Rightarrow$ [Rudin, RCA, 6.6.]
$\mu = \mu^+ - \mu^-$ where $\mu_\pm$ are bounded positive Borel measures ($\mu^\pm := \frac{1}{2}(|\mu| \pm \mu)$). If $\mu^+ \neq 0$ then $c^+ = \int d\mu^+ > 0 \Rightarrow \frac{1}{c^+}\mu^+$ is a Borel probability measure
$\Rightarrow \exists v \in V$ s.t. $\Lambda v = \frac{1}{c^+} \int d\mu^+ \Lambda \circ f \quad \forall \Lambda \in V^*$.

$\Rightarrow$ for $v^+ := c^+ v$ we have $\Lambda v^+ = \int d\mu^+ \Lambda \circ f \quad \forall \Lambda \in V^*$
If $\mu^+ = 0$ this if true for $v^+ = 0$. (RHS is then zero.)
Similarly, $\exists v^- \in V$ s.t. $\Lambda v^- = \int d\mu^- \Lambda \circ f \quad \forall \Lambda \in V^*$
$\Rightarrow \Lambda(v^+ - v^-) = \Lambda v^+ - \Lambda v^- = \int d\mu^+ \Lambda \circ f - \int d\mu^- \Lambda \circ f$
$= \int d\mu \, \Lambda \circ f \Rightarrow v = v^+ - v^-$ satisfies $(*)$. $\forall \Lambda \in V^*$.
$\therefore$ The theorem holds for $\mathbb{K} = \mathbb{R}$.

Suppose $\mathbb{K} = \mathbb{C}$, and μ is a bounded real Borel measure.
If $\Lambda \in V^* \overset{12.3.}{\Rightarrow}$ $\mathrm{Re}\,\Lambda$ and $\mathrm{Im}\,\Lambda = \mathrm{Re}(-i\Lambda)$ are contin. real-lin. functionals $\Rightarrow \exists v \in V$ s.t. $\binom{\mathrm{Re}}{\mathrm{Im}} \Lambda v = \int d\mu \binom{\mathrm{Re}}{\mathrm{Im}} \Lambda \circ f \quad \forall \Lambda \in V^*$.
$\Rightarrow \Lambda v = \int d\mu \, \Lambda \circ f \quad \forall \Lambda \Rightarrow$ Theorem holds. $^{\circledast}$

Finally, if $\mathbb{K} = \mathbb{C}$ and μ is a bounded complex Borel measure, we can write $\mu = \mu_1 + i\mu_2$ where μ_1, μ_2 are bounded real Borel measures.
As above, $\Rightarrow (*)$ holds for some $v = v_1 + i v_2 \in V$ $\square$

$\circledast$ Also, if μ is a prob. measure, $v \in \overline{\mathrm{Hull}(f(Q))}$.

15.3. Corollaries: Suppose μ is a bounded, $\mathbb{K}$-valued Borel measure on a compact Hausdorff set Q.

a) If $\mathcal{F}$ is a Fréchet space and $f : Q \to \mathcal{F}$ is continuous, then $\exists!\, u \in \mathcal{F}$ s.t.

$$u = \int_Q d\mu\, f, \text{ as a VVI.}$$

b) If $\mathcal{B}$ is a Banach space and $f : Q \to \mathcal{B}$ is continuous, then $\|f\|$ is Borel measurable, $\exists!$ VVI $\int_Q d\mu f \in \mathcal{B}$, and

$$(*) \quad \left\| \int_Q d\mu\, f \right\| \leq \int_Q d|\mu|\, \|f\| < \infty.$$

c) If V_1, V_2 are topol. vect. spaces whose duals separate points on them, $T : V_1 \to V_2$ is continuous and linear, $f : Q \to V_1$ is continuous, and $\overline{\text{Hull}(f(Q))}$ is compact, then $\exists!$ VVIs $\int_Q d\mu\, f \in V_1$, $\int_Q d\mu\, T \circ f \in V_2$ and

$$T\left(\int_Q d\mu\, f\right) = \int_Q d\mu\, T \circ f$$

d) If $\mathcal{F}_1, \mathcal{F}_2$ are Fréchet spaces, $T : \mathcal{F}_1 \to \mathcal{F}_2$ is contin. and linear, $f : Q \to \mathcal{F}_1$ is continuous, then

$$T(\underbrace{\textstyle\int_Q d\mu\, f}_{\in \mathcal{F}_1}) = \underbrace{\textstyle\int_Q d\mu\, T \circ f}_{\in \mathcal{F}_2}$$

Proof 'a)' $\mathcal{F}$ = Fréchet $\Rightarrow$ locally convex $\Rightarrow$ $\mathcal{F}^*$ separates points on $\mathcal{F}$ (12.7.a) Also f contin. $\Rightarrow$ $f(Q)$ compact in $\mathcal{F}$ $\Rightarrow$ $\overline{\text{Hull}(f(Q))}$ is compact by 14.12. Thus can apply 15.2. $\Rightarrow \exists!\, u \in \mathcal{F}$. ∎

b) $\mathcal{B}$ = Banach $\Rightarrow$ Fréchet, and thus by a), $\exists!\, u_0 \in \mathcal{B}$ s.t. $u_0 = \int_Q d\mu\, f$. By 12.4. b) $\exists \Lambda \in \mathcal{B}^*$ s.t. $\|u_0\| = \Lambda u_0$ and $|\Lambda u| \leq \|u\|$ $\forall u \in \mathcal{B}$ [RCA, 6.12.]
But then

$$\|u_0\| = \Lambda u_0 \overset{\text{VVI}}{=} \int_Q \mu(dx)\, \Lambda(f(x)) \leq \int_Q |\mu|(dx)\, |\Lambda(\overset{\in \mathcal{B}}{f(x)})|$$
$$\leq \int_Q d|\mu|\, \|f\|, \text{ where } \|f\| : Q \to \mathbb{R} \text{ is continuous}$$