13.9. Definition: (Weak*-topology on $\mathcal{N}^*$)

Assume $\mathcal{N}$ is a topological vector space, and denote $\mathcal{K} := \{ f_x : \mathcal{N}^* \to \mathbb{K} \mid x \in \mathcal{N} \}$ where $f_x(\Lambda) := \Lambda x \ \forall x \in \mathcal{N}, \Lambda \in \mathcal{N}^*$. The $\mathcal{K}$-weak topology on $\mathcal{N}^*$ is called the weak*-topology, denoted $\mathcal{T}_{w*}$.

13.10. Proposition: Consider a topol. vect. space $\mathcal{N}$.

(a) $\mathcal{K}$ is a separating vector space of linear functionals on $\mathcal{N}^*$.

(b) The weak*-topology makes $\mathcal{N}^*$ into a locally convex topol. vect. space whose dual is $\mathcal{K}$.

Proof: a) If $x \in \mathcal{N}$, then $f_x(\alpha\Lambda_1 + \beta\Lambda_2) = \alpha\Lambda_1 x + \beta\Lambda_2 x$ $= \alpha f_x(\Lambda_1) + \beta f_x(\Lambda_2) \Rightarrow f_x$ in lin. functional.
Also $f_0(\Lambda) = \Lambda 0 = 0 \ \forall \Lambda \in \mathcal{N}^* \Rightarrow f_0 = 0 \in \mathcal{K}$. If $\alpha, \beta \in \mathbb{K}$, $x_1, x_2 \in \mathcal{N} \Rightarrow (\alpha f_{x_1} + \beta f_{x_2})(\Lambda) := \alpha f_{x_1}(\Lambda) + \beta f_{x_2}(\Lambda)$
$= \alpha \Lambda x_1 + \beta \Lambda x_2 = \Lambda(\alpha x_1 + \beta x_2) = f_{\alpha x_1 + \beta x_2}(\Lambda) \ \forall \Lambda \in \mathcal{N}^*$.
Since $\alpha x_1 + \beta x_2 \in \mathcal{N} \Rightarrow \alpha f_{x_1} + \beta f_{x_2} \in \mathcal{K}$.
$\therefore \mathcal{K}$ is a vector space. Now if $\Lambda_1, \Lambda_2 \in \mathcal{N}^*$ and $\Lambda_1 \neq \Lambda_2$ $\Rightarrow \exists x \in \mathcal{N}$ s.t. $\Lambda_1 x \neq \Lambda_2 x \Rightarrow f_x(\Lambda_1) \neq f_x(\Lambda_2)$ and $f_x \in \mathcal{K}$. $\therefore \mathcal{K}$ is separating.
b) Follows from a) and theorem 13.1 $\square$

13.11. Theorem (Banach-Alaoglu)

Suppose $\mathcal{N}$ is a topol. vect. space, and $V_0 \in \mathcal{T}_{\mathcal{N}}$ with $0 \in V_0$. Then the set

$$K_{V_0} := \{ \Lambda \in \mathcal{N}^* \mid |\Lambda x| \leq 1 \ \forall x \in V_0 \} \quad (= \text{"Polar" of } V_0)$$

is convex, balanced, and weak*-compact.

Proof: Denote $K := K_{V_0}$. If $\Lambda_1, \Lambda_2 \in K$, $t \in [0,1]$, $x \in V$ then $|(t\Lambda_1 + (1-t)\Lambda_2)(x)| = |t\Lambda_1 x + (1-t)\Lambda_2 x|$ $\le t \cdot 1 + (1-t)\cdot 1 = 1$. $\Rightarrow t\Lambda_1 + (1-t)\Lambda_2 \in K$.
$\therefore$ K is convex. If $\Lambda \in K$, $\alpha \in \mathbb{K}$, and $|\alpha| \le 1$, then $|(\alpha\Lambda)(x)| = |\alpha||\Lambda x| \le 1 \Rightarrow \alpha\Lambda \in K$.
$\therefore$ K is balanced. It remains to prove that K is weak*-compact.

By 4.4. a), for any $x \in V$ $\exists \gamma(x) \in \mathbb{N}_+$ s.t. $x \in \gamma(x) V_0$. Set $D_x := \{\alpha \in \mathbb{K} \mid |\alpha| \le \gamma(x)\}$, $x \in V$. Each D_x is compact in $\mathbb{K}$, and thus $P := \prod_{x \in V} D_x$ is compact in the product topology τ_p on P. (By Tychonoff's theorem; Topo II or [Rudin, FA, p. 368].)
By definition, $g \in P$ is equivalent to $g: V \to \mathbb{K}$ and $|g(x)| \le \gamma(x)$ $\forall x \in V$.

Now if $\Lambda \in K \Rightarrow \Lambda: V \to \mathbb{K}$ and, if $x \in V$, we have $|\Lambda x| = \gamma(x) \left|\Lambda\left(\frac{1}{\gamma(x)} x\right)\right| \le \gamma(x)$ since $\frac{1}{\gamma(x)} x \in V_0$. $\Rightarrow \Lambda \in P$.
Thus $K \subset V^* \cap P$. Let τ denote the topology K inherits from P and τ^* the topology it inherits from τ_{w*}.
We claim that (a) $\tau = \tau^*$ and (b) K is closed subset of P. Then (b) $\Rightarrow$ K is compact under $\tau \overset{a)}{\Rightarrow}$ K is compact under $\tau^* \Rightarrow K \subset V^*$ is weak*-compact. Thus it suffices to prove (a) and (b).

"(a)" Fix $\Lambda_0 \in K$. For $n \in \mathbb{N}_+$, $x_i \in V$, $i = 1, \dots, n$, and $\varepsilon > 0$, consider

$$W_1 := \{\Lambda \in V^* \mid \underbrace{|\Lambda x_i - \Lambda_0 x_i|}_{= f_{x_i}(\Lambda - \Lambda_0)} < \varepsilon \ \forall 1 \le i \le n\}, \text{ and}$$
$$W_2 := \{g \in P \mid |g(x_i) - \Lambda_0 x_i| < \varepsilon \ \forall 1 \le i \le n\}.$$

$\Rightarrow \Lambda_0 \in W_1 = \Lambda_0 + V(n, (f_{x_1}, \dots, f_{x_n}), \varepsilon) \in \mathcal{T}_{w*}$ and $\Lambda_0 \in W_2 = \bigcap_{i=1}^{n} \mathrm{Proj}_{x_i}^{\leftarrow}(B(\Lambda_0 x_i, \varepsilon)) \in \mathcal{T}_p$. If $U \in \tau$, $\Lambda_0 \in U$
$\Rightarrow \exists U' \in \tau_p$ s.t. $U = K \cap U' \Rightarrow \Lambda_0 \in U'$ and $\exists n \in \mathbb{N}_+$, and U_i, $i = 1, \dots, n$, open in $\mathbb{K}$ s.t. $\Lambda_0 \in \bigcap_{i=1}^{n} \mathrm{Proj}_{x_i}^{\leftarrow} U_i \subset U'$.
$\Rightarrow \mathrm{Proj}_{x_i}(\Lambda) := \Lambda(x_i) \in U_i$ $\forall i \Rightarrow$
$\exists \varepsilon > 0$ s.t. $B(\Lambda(x_i), \varepsilon) \subset U_i$ $\forall i \Rightarrow \exists W_2$ as above s.t. $\Lambda_0 \in W_2 \subset U'$. $\Rightarrow K \cap W_2 \subset U$. Obviously, always ($K \subset V^* \cap P$) $K \cap W_1 = K \cap W_2$, and thus we have found $\Lambda_0 \in K \cap W_1 \subset U$ where $K \cap W_1 \in \mathcal{T}^*$ $\therefore \tau \subset \tau^*$.
If $U \in \tau^*$, $\Lambda_0 \in U \Rightarrow \exists W_1$ as above s.t. $\Lambda_0 \in K \cap W_1 \subset U$ $\Rightarrow \Lambda_0 \in K \cap W_2 \subset U$ and $K \cap W_2 \in \tau$ $\therefore \tau^* \subset \tau$ ◪

"(b)" Suppose $g_0 \in \bar{K}^{\tau_P} \subset P$. For $x_1, x_2 \in \mathcal{V}$, $\alpha, \beta \in \mathbb{K}$ and $\varepsilon > 0$, set $x_3 = \alpha x_1 + \beta x_2 \in \mathcal{V}$, and consider

$W_3 := \{ g \in P \mid |g(x_i) - g_0(x_i)| < \varepsilon \ \forall i = 1,2,3 \}$.

$\Rightarrow W_3 = \bigcap_{i=1}^{3} \mathrm{Proj}_{x_i}^{\leftarrow}(B(g_0(x_i), \varepsilon))$ and thus $g_0 \in W_3 \in \mathcal{T}_P$.

$\Rightarrow \exists g \in K \cap W_3 \Rightarrow g$ is linear $\Rightarrow$

$g_0(\alpha x_1 + \beta x_2) - \alpha g_0(x_1) - \beta g_0(x_2)$

$= (g_0 - g)(\alpha x_1 + \beta x_2) - \alpha (g_0 - g)(x_1) - \beta (g_0 - g)(x_2)$

$\Rightarrow |g_0(\alpha x_1 + \beta x_2) - \alpha g_0(x_1) - \beta g_0(x_2)|$

$\leq \varepsilon(1 + |\alpha| + |\beta|) \quad \forall \varepsilon > 0$

$\therefore g_0(\alpha x_1 + \beta x_2) = \alpha g_0(x_1) + \beta g_0(x_2) \quad \forall \alpha, \beta, x_1, x_2$

$\Rightarrow g_0$ is linear map $\mathcal{V} \to \mathbb{K}$. If $x \in V_0$ and $\varepsilon > 0$

$\Rightarrow \exists g \in K$ s.t. $|g(x) - g_0(x)| < \varepsilon$.

$\Rightarrow |g_0(x)| \leq |g(x)| + |g_0(x) - g(x)| < 1 + \varepsilon$.

$\therefore |g_0(x)| \leq 1$. Since V_0 is a neighborhood of 0

$\underset{5.3.d)}{\Rightarrow} g_0$ is continuous. $\therefore g_0 \in K$ $\square$

13.12. Reminder A topological space X is called separable if it has a countable dense subset.

13.13. Proposition Suppose $\mathcal{V}$ is a separable topol. vect. space.

(a) If $K \subset \mathcal{V}^*$ is weak*-compact, then K is metrizable in the weak*-topology.
(Warning! $\mathcal{V}^*$ is typically not metrizable in $\mathcal{T}_{w*}$)

(b) If $V_0 \in \mathcal{T}_{\mathcal{V}}$, $0 \in V_0$, and $\Lambda_n \in \mathcal{V}^*$, $n \in \mathbb{N}_+$, is a sequence for which $|\Lambda_n x| \leq 1 \ \forall x \in V_0, n \in \mathbb{N}_+$, then $\exists$ subsequence $(\Lambda_{n_i})_{i \in \mathbb{N}_+}$ and $\Lambda \in \mathcal{V}^*$ s.t.

$$\Lambda_{n_i} x \xrightarrow{i \to \infty} \Lambda x \quad \forall x \in \mathcal{V}$$

Proof. a) Let $\{x_n\}_{n \in \mathbb{N}_+} \subset \mathcal{V}$ be dense. Set $g_n := f_{x_n}$, i.e. $g_n(\Lambda) = \Lambda x_n \ \forall \Lambda \in \mathcal{V}^*$. If $g_n(\Lambda') = g_n(\Lambda) \ \forall n$ $\Rightarrow \Lambda' = \Lambda$ on the dense set $\{x_n\}$ $\Rightarrow \Lambda' = \Lambda$ everywhere, as both Λ and Λ' are continuous and $\mathbb{K}$ is Hausdorff.

[If $\Lambda|_0 = 0$ and $\Lambda x_0 \neq 0 \Rightarrow \exists U \subset \mathbb{k}$ open s.t. $\Lambda x_0 \in U$, but $0 \notin U \Rightarrow x_0 \in \Lambda^{\leftarrow} U \in \mathcal{T}_N$ and $D \cap \Lambda^{\leftarrow} U = \phi \Rightarrow x_0 \notin \overline{D}$.]

Thus $\{g_n\}$ is a separating family on N^*, and each g_n is weak*-continuous (by definition). $\Rightarrow$ the same is true of $\{\mathrm{Re}\, g_n|_K, \mathrm{Im}\, g_n|_K\}_{n \in \mathbb{N}_+}$. Since K is w*-compact, by Ex. 8.3.b), its weak*-topology is metrizable. ◼

b) By 13.11., the polar K_{V_0} is weak*-compact, and by assumption $\{\Lambda_n\} \subset K_{V_0}$. Now a) $\Rightarrow K_{V_0}$ is compact and metric $\Rightarrow$ sequentially compact (Topo II) $\Rightarrow \exists$ weak*-convergent subsequence $\Lambda_{n_i} \to \Lambda \in K_{V_0} \subset N^*$. $\Rightarrow \forall x \in N: \Lambda_{n_i} x = f_x(\Lambda_{n_i}) \xrightarrow{i\to\infty} f_x(\Lambda) = \Lambda x$. □

<u>13.14. Theorem</u> In a <u>locally convex</u> topol. vect. space N, every weakly bounded set is originally bounded, and vice versa.

<u>Proof.</u> "Vice versa" If $E \subset N$ orig. bounded and $0 \in U \in \mathcal{T}_W$.

$\underset{U \in \mathcal{T}_W}{\Rightarrow} U \in \mathcal{T}_0 \Rightarrow \exists s > 0$ s.t. $E \subset tU \;\forall t > s$.

$\Rightarrow$ E is weakly bounded.

"$\Rightarrow$" Suppose $E \subset N$ is weakly bounded and $0 \in U \in \mathcal{T}_0$. As N is locally convex, it follows from 4.9, that $\exists U' \in \mathcal{T}_0$ s.t. $0 \in U'$, U' is convex, and $\overline{U'} \subset U$. By 4.12.b), $\exists V_0 \in \mathcal{T}_0$ s.t. $0 \in V_0$, V_0 is balanced and convex, and $\overline{V_0} \subset \overline{U'} \subset U$. Let $K = K_{V_0}$ as in 13.11.

Consider $C := \{x \in N \mid |\Lambda x| \leq 1 \;\; \forall \Lambda \in K\}$.

If $x \in V_0$ & $\Lambda \in K \Rightarrow |\Lambda x| \leq 1$ (by def.). Thus $V_0 \subset C$.

Since $C = \bigcap_{\Lambda \in K} \Lambda^{\leftarrow} \overline{B}(0,1)$, where $\overline{B}(0,1) = \{y \in \mathbb{k} \mid |y| \leq 1\}$ is closed and each $\Lambda \in K$ is continuous $N \to \mathbb{k}$, C is closed and thus $\overline{V_0} \subset C$. By 4.11., $\overline{V_0}$ is convex, balanced and $\mathcal{T}_0$-closed, and thus by 12.7.c) for any $x_0 \in N \setminus \overline{V_0}$ $\exists \Lambda \in N^*$ s.t. $|\Lambda x| \leq 1 \;\forall x \in \overline{V_0}$ and $\Lambda x_0 > 1 \Rightarrow \Lambda \in K$ and $\Lambda x_0 > 1 \Rightarrow x_0 \notin C$. Therefore, $C \subset \overline{V_0}$ $\therefore$ $C = \overline{V_0}$.

Suppose $\Lambda \in N^*$. Since E is weakly bounded, by 13.5, c) $\Rightarrow \Lambda|_E$ is bounded $\Rightarrow \exists \gamma(\Lambda) < \infty$ s.t. $|\Lambda x| \leq \gamma(\Lambda) \;\; \forall x \in E$. (Ex. 4.2.a)

Thus if we define $\Gamma := \{ f_x : \mathcal{V}^* \to \mathbb{k} \mid x \in E \}$, where $f_x(\Lambda) = \Lambda x$ as before, then each f_x is lin. and w^*-continuous (13.10.) and the orbit of any $\Lambda \in \mathcal{V}^*$, $\{ f_x(\Lambda) \mid x \in E \} = \{ \Lambda x \mid x \in E \}$, is bounded in $\mathbb{k}$. Since by Banach-Alaoglu $K \subset \mathcal{V}^*$ is w^*-compact and convex, we can apply Theorem 10.11. and conclude that $\exists \bar{\gamma} > 0$, s.t. for any $x \in E$, $\Lambda \in K$, $|\Lambda x| = |f_x(\Lambda)| \le \bar{\gamma} \Rightarrow |\Lambda(\bar{\gamma}^{-1} x)| \le 1$. Thus $\forall x \in E$, $\bar{\gamma}^{-1} x \in C = \bar{V}_0 \Rightarrow$ if $t > \bar{\gamma}$, then $t^{-1} x = \frac{\bar{\gamma}}{t} \bar{\gamma}^{-1} x$ $\in \underbrace{\frac{\bar{\gamma}}{t}}_{<1} \bar{V}_0 \underset{\bar{V}_0 \text{ balanced}}{\subset} \bar{V}_0 \subset U$ $\therefore$ $E \subset tU$ $\forall t > \bar{\gamma}$.

Therefore, E is originally bounded. $\square$

13.15. Corollary Suppose $\mathcal{V}$ is a normed space. If $E \subset \mathcal{V}$ is s.t. for any $\Lambda \in \mathcal{V}^*$, $\sup_{x \in E} |\Lambda x| < \infty$, then $\exists M < \infty$ s.t. $\|x\| \le M \; \forall x \in E$.

Proof. $\mathcal{V}$ normed $\Rightarrow$ locally convex. If $E \subset \mathcal{V}$ is s.t. $\forall \Lambda \in \mathcal{V}^*$ $\sup_{x \in E} |\Lambda x| < \infty \Rightarrow \Lambda|_E$ is bounded $\forall \Lambda \in \mathcal{V}^*$ $\overset{13.5.c)}{\Rightarrow}$ E is weakly bounded $\overset{13.14.}{\Rightarrow}$ E is originally (=norm)-bounded $\overset{Ex. 4.2.a)}{\Rightarrow}$ $\exists M < \infty$ s.t. $\sup_{x \in E} \|x\| \le M$ $\square$

14. Compact convex sets and Krein-Milman Theorem

14.1. Definition: Let X be a vector space, and $K \subset X$.

(a) If $S \subset K$, $S \neq \emptyset$, and

$x, y \in K$, $0 < t < 1$, with $tx + (1-t)y \in S$ always implies $x, y \in S$

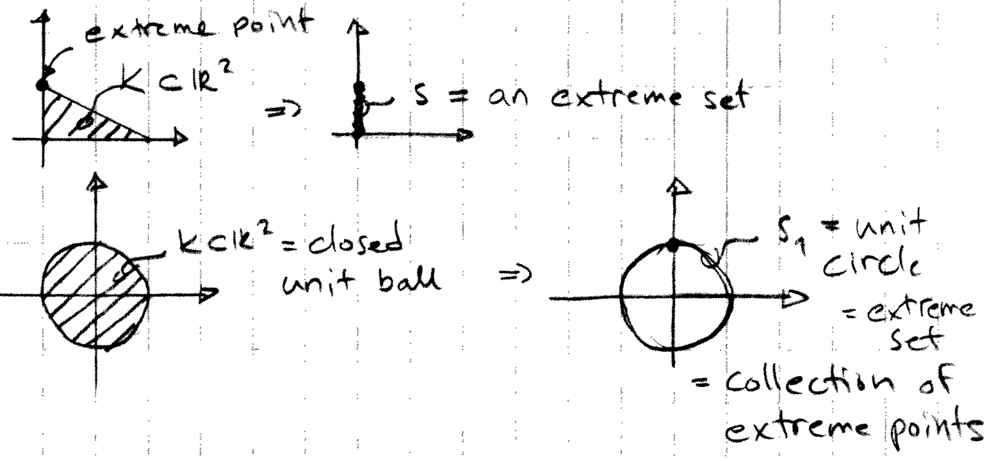
then S is an extreme set of K.

(b) If $x_0 \in K$ is s.t. $\{x_0\}$ is an extreme set of K, then x_0 is called an extreme point of K.

* Denote ℓ_{xy} = line segment connecting x with $y \subset X$. Geometrically, $\emptyset \neq S \subset K$ is an extreme set iff no internal point of ℓ_{xy}, $x, y \in K$, can belong to S, unless both ends belong to S.

* Thus $x_0 \in K$ is an extreme point iff x_0 intersects no ℓ_{xy} with $x, y \in K \setminus \{x_0\}$.
Alternatively, $x_0 \in K$ is an extreme point iff it cannot be written as a proper convex combination of two distinct elements of K.

Examples:



14.2. Theorem (Krein-Milman)

Suppose V is a topol. vect. space on which V^* separates points. If $K \subset V$ is compact and convex, then

$$K = \overline{\mathrm{Hull}(S_0)}, \quad \text{where } S_0 := \{x \in K \mid x \text{ is an extreme point of } K\}.$$

* Note that then $S_0 = \emptyset \Rightarrow K = \emptyset$.
* The results says that any point in K can be approximated by finite lin. combinations of extreme points of K.

For the proof, we need one more version of the Hahn-Banach theorem:

14.3. Theorem Suppose V is a topol. vect. space on which V^* separates points.
If $A, B \subset V$ are nonempty, disjoint, compact and convex, then $\exists \Lambda \in V^*$ s.t.

$$\sup_{x \in A} (\mathrm{Re}\, \Lambda x) < \inf_{y \in B} (\mathrm{Re}\, \Lambda y).$$

Proof: By 13.4, now $V_w = V$ with the weak topology is a locally convex t.v.s. and $(V_w)^* = V^*$.
As also $\tau_W \subset \tau_0$, both A, B are compact in V_W
$\Rightarrow$ closed in V_W (since it is a Hausdorff space).
Thus by 12.5.b) $\Rightarrow \exists \Lambda \in (V_W)^* = V^*$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ s.t. $\mathrm{Re}\, \Lambda x < \gamma_1 < \gamma_2 < \mathrm{Re}\, \Lambda y \quad \forall x \in A, y \in B$
$\Rightarrow \sup_{x \in A} (\mathrm{Re}\, \Lambda x) \leq \gamma_1 < \gamma_2 \leq \inf_{y \in B} (\mathrm{Re}\, \Lambda y)$. $\square$

* Note: Unlike in 12.5.b), here V is not assumed to be locally convex.

14.4. Lemma Suppose V is a topol. vect. space on which V^* separates points. If $K \subset V$ and $S_1 \subset K$ is a compact extreme set of K, then S_1 contains an extreme point of K.

Proof. Set $\mathcal{P} := \{ S \subset S_1 \mid S \text{ is a compact extreme set of } K \}$
$\Rightarrow S_1 \in \mathcal{P}$ and thus $\mathcal{P} \neq \emptyset$.

"a)" Suppose $\mathcal{C} \subset \mathcal{P}$, $\mathcal{C} \neq \emptyset$. Denote $S_{\mathcal{C}} := \bigcap_{S \in \mathcal{C}} S$. As each $S \in \mathcal{C}$ is compact and V is Hausdorff
$\Rightarrow$ every $S \in \mathcal{C}$ is closed $\Rightarrow S_{\mathcal{C}}$ is closed. If $S \in \mathcal{C}$
$\Rightarrow S$ compact & $S_{\mathcal{C}} \subset S$. Thus $S_{\mathcal{C}}$ is compact.
Consider $x, y \in K$, $0 < t < 1$ and $S \in \mathcal{C}$. If $tx + (1-t)y \in S_{\mathcal{C}}$
$\Rightarrow tx + (1-t)y \in S \underset{S \text{ extreme set}}{\Rightarrow} x, y \in S$. Since S is arbitrary,
we have $tx + (1-t)y \in S_{\mathcal{C}} \Rightarrow x, y \in \bigcap_{S \in \mathcal{C}} S = S_{\mathcal{C}} \subset K$

Thus now either $S_{\mathcal{C}} = \emptyset$ or $S_{\mathcal{C}} \in \mathcal{P}$.

b) Suppose $S \in \mathcal{P}$ and $\Lambda \in N^*$. Then $\mathrm{Re}\,\Lambda : N \to \mathbb{R}$ is contin. and $S \subset N$ is compact $\Rightarrow$ $\exists\, \mu := \max_{x \in S} \mathrm{Re}\,\Lambda x \in \mathbb{R}$ and

$S(S,\Lambda) := \{ x \in S \mid \mathrm{Re}\,\Lambda x = \mu \} \neq \emptyset$. Since $S(S,\Lambda) \subset S_1$ = compact and $S(S,\Lambda) = ((\mathrm{Re}\,\Lambda)^{-1}\{\mu\}) \cap S$ is closed $\Rightarrow$ $S(S,\Lambda)$ is compact and a subset of S_1. Suppose then $x, y \in K$, $0 < t < 1$ and $z := tx + (1-t)y \in S(S,\Lambda)$. As then $z \in S$ = extreme set $\Rightarrow$ $x, y \in S$ $\Rightarrow$ $\mathrm{Re}\,\Lambda x,\ \mathrm{Re}\,\Lambda y \leq \mu$. Now by def.
$\mu = \mathrm{Re}\,\Lambda z = t\, \mathrm{Re}\,\Lambda x + (1-t)\,\mathrm{Re}\,\Lambda y \leq \underbrace{(t+1-t)}_{=1} \max(\mathrm{Re}\,\Lambda x, \mathrm{Re}\,\Lambda y)$
$\leq \mu$ $\Rightarrow$ max $= \mu$ $\Rightarrow$ also min $= \mu$. ($0 < t < 1$).
therefore, $\mathrm{Re}\,\Lambda x = \mu = \mathrm{Re}\,\Lambda y$ $\Rightarrow$ $x, y \in S(S,\Lambda)$.
$\therefore$ $S(S,\Lambda)$ is an extreme set of K. $\therefore$ $S(S,\Lambda) \in \mathcal{P}$.

Now inclusion "$\subset$" forms a partial order on $\mathcal{P}$.
[Proof: $a, b, c \in \mathcal{P}$ $\Rightarrow$ $a \subset a$, if $a \subset b$ & $b \subset a$ then $a = b$, and if $a \subset b$ & $b \subset c$ $\Rightarrow$ $a \subset c$. ∎] Suppose $\Theta \subset \mathcal{P}$ is totally ordered and non-empty. Define $S_\Theta := \bigcap_{S \in \Theta} S$ as in "a)" and choose $S' \in \Theta$ ($\neq \emptyset$).
Since Θ is totally ordered, for any finite non-empty collection $A \subset \Theta$ we have $\bigcap_{S \in A} S \in A$. [Proof: Induction in $|A| = N$. $N=1$ obvious. If $|A| = N+1$, $S'' \in A$ $\Rightarrow$ $\bigcap_{S \in A} S = S'' \cap (\bigcap_{S \in A \setminus S''} S) \overset{\text{ind. assumpt.}}{=} S'' \cap S'''$ where $S' \subset S'''$ or $S''' \subset S''$. ∎]
Since each $S \in \Theta$ is nonempty and compact, this finite intersection property implies $S_\Theta \neq \emptyset$, as on p. 55
[Proof: $S_\Theta \subset S'$ = compact. If $S_\Theta = \emptyset$ $\Rightarrow$ $N = \bigcup_{S \in \Theta} S^c$
$\Rightarrow$ $\{S^c\}_{S \in \Theta}$ is an open cover of S'
$\Rightarrow$ $\exists A \subset \Theta$, $|A| < \infty$, s.t. $S' \subset \bigcup_{S \in A} S^c = (\bigcap_{S \in A} S)^c = (S'')^c$ for some $S'' \in \Theta$.
$\Rightarrow$ $S'' \cap S' = \emptyset$ ↯ ∎]
Then, by "a)", necessarily $S_\Theta \in \mathcal{P}$, and since $S_\Theta \subset S$ $\forall S \in \Theta$, S_Θ is a lower bound for Θ.
By Zorn's lemma, there is then a minimal element S_0 in $\mathcal{P}$. By "b)", for any $\Lambda \in N^*$ we have $S(S_0,\Lambda) \subset S_0$ and $S(S_0,\Lambda) \in \mathcal{P}$ $\underset{S_0 \text{ minimal}}{\Rightarrow}$ $S(S_0,\Lambda) = S_0$ $\Rightarrow$ $\forall x \in S_0$: $\mathrm{Re}\,\Lambda x$ = const.

If $\mathbb{K} = \mathbb{C}$, then $(-i)\Lambda \in N^*$ $\Rightarrow$ $\forall x \in S_0$: $\mathrm{Re}[(-i)\Lambda x] = \mathrm{Im}\,\Lambda x$ = const.
$\therefore$ $\Lambda|_{S_0}$ is constant, and if $\mathbb{K} = \mathbb{R}$ this follows directly from $\Lambda = \mathrm{Re}\,\Lambda$. Since $S_0 \in \mathcal{P}$ $\Rightarrow$ $S_0 \neq \emptyset$, and if $x, y \in S_0$

and $x \neq y$ then $\exists \Lambda \in V^*$ s.t. $\Lambda x \neq \Lambda y \Rightarrow \Lambda|_{S_0}$ not const. ↯
Therefore $\exists x_0 \in S_1$ s.t. $S_0 = \{x_0\}$ = extreme set of K
$\Rightarrow$ x_0 = extreme point of K □

<u>Proof of 14.2.</u>: If $K = \emptyset \Rightarrow S_0 = \emptyset \Rightarrow \overline{Hull(S_0)} = \emptyset = K$, and the result holds. Assume thus $K \neq \emptyset$.
Since then K is a compact extreme set of K, Lemma 14.4. $\Rightarrow \exists x \in S_0 \Rightarrow S_0 \neq \emptyset$. Set $H := Hull(S_0) \neq \emptyset$. Since K is convex and $S_0 \subset K \overset{Ex.4.5.}{\Rightarrow} H \subset K \Rightarrow \overline{H} \subset \overline{K} = K$ since K is compact and V is Hausdorff. $\therefore \overline{H}$ = compact and convex.
Suppose that $\exists x_1 \in K \setminus \overline{H}$. Applying 14.3. to the pair $\{x_1\}$ and $\overline{H} \Rightarrow \exists \Lambda \in V^*$ s.t. $Re\,\Lambda x < Re\,\Lambda x_1 \;\; \forall x \in \overline{H}$.
Set then $S_1 := K$ and define $\mathcal{P}$ and $\zeta(S, \Lambda)$, $S \in \mathcal{P}$, as in the proof of Lemma 14.4. By "b)" of that proof, then $\zeta(K, \Lambda) \in \mathcal{P}$ since $K \in \mathcal{P}$, $\Rightarrow \zeta(K,\Lambda)$ is a compact extreme set of $K \underset{\text{Lemma 14.4.}}{\Rightarrow} \exists x_0 \in S_0$ s.t. $x_0 \in \zeta(K,\Lambda) \Rightarrow x_0 \in \overline{H} \cap \zeta(K,\Lambda)$.

$$\Rightarrow \quad Re\,\Lambda x_0 < Re\,\Lambda x_1 \leq \sup_{x \in K} Re\,\Lambda x = \mu(K,\Lambda) \overset{!}{=} Re\,\Lambda x_0 . \; ↯$$

Thus $K \setminus \overline{H} = \emptyset$. $\therefore K = \overline{H}$ □

<u>14.5. Theorem</u>: Suppose V is a <u>locally convex</u> topol. vect. space, and $K \subset V$ is compact.
Then $K \subset \overline{Hull(S_0)}$ for $S_0 := \{x$ is an extreme point of $K\}$.

<u>Proof.</u> V locally convex $\overset{12.7.a)}{\Rightarrow} V^*$ separates points on V. If $K = \emptyset$ $\Rightarrow S_0 = \emptyset$ and $K \subset \emptyset$. Assume $K \neq \emptyset \Rightarrow K$ is an compact extreme set of $K \Rightarrow K \in \mathcal{P}$ with $S_1 = K$ in the proof of Lemma 14.4. $\Rightarrow S_0 \neq \emptyset$. Denote $H := Hull(S_0) \neq \emptyset$. Suppose $\exists x_1 \in K \setminus \overline{H}$.
$\underset{12.5.b)}{\Rightarrow} \exists \Lambda \in V^*$ s.t. $Re\,\Lambda x_1 < Re\,\Lambda x \;\; \forall x \in \overline{H}$. Set $\Lambda_0 := -\Lambda \in V^*$

$\Rightarrow$ $Re\,\Lambda_0 x_1 > Re\,\Lambda_0 x \;\; \forall x \in \overline{H}$. Since $K \in \mathcal{P}$, also $\zeta(K, \Lambda_0) \in \mathcal{P}$
$\overset{14.4.}{\Rightarrow} \exists x_0 \in S_0 \cap \zeta(K, \Lambda_0) \Rightarrow Re\,\Lambda_0 x_0 \underset{S_0 \subset \overline{H}}{<} Re\,\Lambda_0 x_1 \underset{x_1 \in K,\, x_0 \in \zeta(K,\Lambda_0)}{\leq} Re\,\Lambda_0 x_0$ ↯.
$\therefore K \setminus \overline{H} = \emptyset \Rightarrow K \subset \overline{H}$ □

<u>14.6. Corollary</u>: Under the assumptions of thrm 14.5. we also have $\overline{Hull(K)} = \overline{Hull(S_0)}$.
<u>Proof.</u> $S_0 \subset K \Rightarrow Hull(S_0) \subset Hull(K) \Rightarrow \overline{Hull(S_0)} \subset \overline{Hull(K)}$.
By 14.5. also $K \subset \overline{Hull(S_0)} \Rightarrow \overline{Hull(K)} \subset \overline{Hull(S_0)}$ □