12.6. Lemma Let $\mathcal{V}$ be a topol. vect. space. If $\Lambda \in \mathcal{V}^*$ and $\Lambda \neq$ constant, then Λ is an open mapping.

Proof Now $\exists x_0 \in \mathcal{V}$ s.t. $\Lambda x_0 \neq \Lambda 0 = 0$. Consider $V \in \mathcal{T}_{\mathcal{V}}$, and $x \in V$. By Thrm. 4.12. $\exists U_0 \in \mathcal{T}_{\mathcal{V}}$ s.t. $0 \in U_0 \subset V - x$ and U_0 is balanced. By 4.14 a), $\exists n_0 \in \mathbb{N}_+$ s.t. $x_0 \in n_0 U_0$.
Then, if $y \in B(\Lambda x, \frac{|\Lambda x_0|}{n_0}) \Rightarrow \exists \alpha \in \mathbb{K}$ with $|\alpha| \leq 1$, s.t.
$y = \Lambda x + \alpha \frac{\Lambda x_0}{n_0}$
$= \Lambda(x + \frac{\alpha}{n_0} x_0) \in \Lambda(x + \alpha U_0) \subset \Lambda(x + U_0) \subset \Lambda(V)$.
$\therefore B(\Lambda x, \underbrace{n_0^{-1}|\Lambda x_0|}_{>0}) \subset \Lambda(V)$. Thus $\Lambda(V)$ is open. $\square$

To complete the proof of 12.5. we need to consider the case $\mathbb{K} = \mathbb{C}$. However, as $\mathcal{V}$ is then also a real vector space, we can apply the previous result and conclude that there is a real-linear $u : \mathcal{V} \to \mathbb{R}$ which is continuous and for which the desired inequality holds. However, by Corollary 12.2.' then $\exists! \Lambda \in \mathcal{V}^*$ s.t. $u = \operatorname{Re} \Lambda$, and thus this Λ has all the required properties. $\square$

12.7. Corollaries:

Consider a <u>locally convex</u> topol. vect. space $\mathcal{V}$.

(a) $\mathcal{V}^*$ separates points on $\mathcal{V}$:
For any $x_1, x_2 \in \mathcal{V}$, $x_1 \neq x_2$, there is $\Lambda \in \mathcal{V}^*$ s.t. $\Lambda(x_1) \neq \Lambda(x_2)$.

(b) If $M \subset \mathcal{V}$ is a <u>subspace</u> and $x_0 \in \mathcal{V} \setminus \overline{M}$, then $\exists \Lambda \in \mathcal{V}^*$ s.t. $\Lambda x_0 = 1$ but $\Lambda x = 0$ $\forall x \in M$.

(c) $\to$

Proof: "a)" Apply 12.5.b) to $A = \{x_1\}$, $B = \{x_2\}$ ($\Rightarrow A \cap B = \emptyset$.)
"b)" Set $A = \{x_0\}$ and $B = \overline{M}$. Then $A \cap B = \emptyset$, $\overline{M}$ is a subspace $\Rightarrow$ convex, and A is compact. Thus by 12.5.b) $\Rightarrow \exists \Lambda \in \mathcal{V}^*$ s.t. $\Lambda(A) = \{\Lambda x_0\}$ and $\Lambda(\overline{M})$ are disjoint $\Rightarrow \Lambda x_0 \notin \Lambda(M) =$ subspace of $\mathbb{K}$.
$\Rightarrow \Lambda(M) = \{0\}$ and $\Lambda x_0 \neq 0$. Therefore $\frac{1}{\Lambda x_0} \Lambda \in \mathcal{V}^*$ has the required properties. $\square$

* Often one can classify V^* explicitly; for instance, $(L^p)^* = L^q$, $1 < p < \infty$, $q = \frac{p}{p-1}$ and $(L^1)^* = L^\infty$

* Then 12.7. b) offers a way of proving that a certain vector $x_0 \in V$, which might be very complicated in general, actually can be approximated by some nice functions forming a subspace M: If one can check that for every $\Lambda \in V^*$ with $\Lambda|_M = 0$ one has $\Lambda(x_0) = 0$, then $x_0 \in \overline{M}$.

(c) Suppose $B \subset V$ is convex, balanced, and closed. If $x_0 \in V \setminus B$, then $\exists \Lambda \in V^*$ s.t. $|\Lambda x| \leq 1$ $\forall x \in B$ but $\Lambda x_0 > 1$.

Proof. Set $A = \{x_0\}$ $\Rightarrow$ $A \cap B = \emptyset$ and can apply 12.5. b).
$\Rightarrow \exists \Lambda \in V^*$ s.t. $\operatorname{Re} \Lambda x_0 < \gamma_1 < \gamma_2 < \operatorname{Re} \Lambda x$ $\forall x \in B$.
As B is balanced $\Rightarrow 0 \in B \Rightarrow \operatorname{Re} \Lambda x_0 < \gamma_1 < 0$.
$\Rightarrow -\operatorname{Re} \Lambda x_0 > -\gamma_1 > 0$. $\Rightarrow \frac{1}{\gamma_1} \operatorname{Re} \Lambda x_0 > 1 > \frac{1}{\gamma_1} \operatorname{Re} \Lambda x$.

As $\Lambda x_0 \neq 0$ can define $\alpha = \dfrac{\operatorname{Re} \Lambda x_0}{\gamma_1 \Lambda x_0}$ and $\tilde{\Lambda} = \alpha \Lambda \in V^*$.

$\Rightarrow \tilde{\Lambda} x_0 = \alpha \Lambda x_0 = \frac{\operatorname{Re} \Lambda x_0}{\gamma_1} > 1$ and for $x \in B$ $\exists \beta_x \in \mathbb{K}$ with $|\beta_x| = 1$ s.t. $|\tilde{\Lambda} x| = \beta_x \tilde{\Lambda} x = \tilde{\Lambda}(\beta_x x)$
$= \operatorname{Re} \tilde{\Lambda}(\beta_x x) = \frac{1}{\gamma_1} \operatorname{Re} \Lambda \Big(\underbrace{\frac{\operatorname{Re} \Lambda x_0}{\Lambda x_0} \beta_x}_{|\,| \leq 1} x\Big) < 1$, since B is balanced.
$\Rightarrow \tilde{\Lambda} \in V^*$, $|\tilde{\Lambda} x| \leq 1$ $\forall x \in B$ and $\tilde{\Lambda} x_0 > 1$. $\square$

12.8. Theorem (Continuous extensions of functionals)

Let V be a locally convex topol. vect. space. If $M \subset V$ is a subspace, and $f : M \to \mathbb{K}$ is contin. and linear, then $\exists \Lambda \in V^*$ s.t. $\Lambda|_M = f$.

Proof. If $f = 0 \Rightarrow \Lambda = 0 \in V^*$ is an extension. Assume thus $f \neq 0 \Rightarrow \exists x_0 \in M$ s.t. $f(x_0) = 1$. Set $M_0 = \ker f \subset M$. Since f is contin. $\Rightarrow M_0$ is closed in M $\Rightarrow \exists U_0 \in \mathcal{T}_V$ s.t. $M \setminus M_0 = U_0 \cap M \Rightarrow x_0 \in U_0$ and $U_0 \cap M_0 = \emptyset \Rightarrow x_0 \notin \overline{M_0}$ (closure in V)

Thus can apply Corollary 12.7.b) $\Rightarrow \exists \Lambda \in V^*$ s.t. $\Lambda x_0 = 1$ and $\Lambda x = 0 \ \forall x \in \overline{M}_0$. (Note that M_0 = subspace $\Rightarrow \overline{M}_0$ = subspace.) Then if $x \in M \Rightarrow f(x - f(x)x_0) = 0$ $\Rightarrow x - f(x)x_0 \in M_0 \Rightarrow 0 = \Lambda(x - f(x)x_0) = \Lambda x - f(x)\underbrace{\Lambda x_0}_{=1}$ $\Rightarrow \Lambda x = f(x)$. Thus $\Lambda|_M = f$. □

13. Weak and weak* topologies

* Recall the $\mathcal{K}$-weak topology defined in 3.5. which was a way of inducing a topology on S using a given topology on X so that each of the maps $S \to X$ in $\mathcal{K}$ is continuous. The following theorem shows that weak topologies induced by linear functionals are typically very nice:

13.1. Theorem Suppose X is a vector space and X' is a collection of linear functionals on X such that X' is a vector space and separates points on X. Then the X'-weak topology makes X into a locally convex topological vector space for which $X^* = X'$.

* Explicitly, the assumptions on X' are:
  (i) $\Lambda \in X' \Rightarrow \Lambda : X \to \mathbb{K}$ and Λ is linear
  (ii) $\Lambda_1, \Lambda_2 \in X'$, $\alpha, \beta \in \mathbb{K} \Rightarrow \alpha\Lambda_1 + \beta\Lambda_2 \in X'$
  (iii) $0 \in X'$
  (iv) $x_1, x_2 \in X$, $x_1 \neq x_2 \Rightarrow \exists \Lambda \in X'$ s.t. $\Lambda x_1 \neq \Lambda x_2$.

  Note also that the X'-weak topology is defined using the standard (vector) topology of $\mathbb{K}$.

For the proof, we need two lemmas

13.2. Lemma: Let $\mathcal{K}$ be a collection of maps $S \to X$ and X have a Hausdorff topology. If $\mathcal{K}$ separates points on X, then the $\mathcal{K}$-weak topol. on S is also Hausdorff.

Proof. Suppose $x_1, x_2 \in S$ with $x_1 \neq x_2$. By assumption, $\exists f \in \mathcal{K}$ s.t. $f(x_1) \neq f(x_2)$. Since $\mathfrak{X}$ is Hausdorff $\Rightarrow \exists U_1, U_2 \in \mathcal{J}_{\mathfrak{X}}$ s.t. $f(x_1) \in U_1$, $f(x_2) \in U_2$ and $U_1 \cap U_2 = \emptyset$. Set $V_1 := f^{\leftarrow}(U_1)$, $V_2 := f^{\leftarrow}(U_2)$. Then $x_1 \in V_1$, $x_2 \in V_2$, and V_1, V_2 are ($\mathcal{K}$-weak) open. In addition, $V_1 \cap V_2 = \bigcap_{i=1,2} f^{\leftarrow}(U_i) = f^{\leftarrow}(U_1 \cap U_2) = \emptyset$. $\therefore$ $\mathcal{K}$-weak is Hausd. $\square$

13.3. Lemma: Suppose Λ_n, $n = 0, 1, \dots, N$, $N \in \mathbb{N}_+$, are linear functionals on a vector space $\mathfrak{X}$. Let $M := \{ x \in \mathfrak{X} \mid \Lambda_i x = 0 \ \forall i = 1, \dots, N \}$. Then the following statements are equivalent:

(a) $\exists \alpha_i \in \mathbb{K}$, $i = 1, 2, \dots, N$, s.t. $\Lambda_0 = \sum_{i=1}^{N} \alpha_i \Lambda_i$

(b) $\exists \gamma \in \mathbb{R}$ s.t. $|\Lambda_0 x| \leq \gamma \max_{i=1,\dots,N} |\Lambda_i x| \quad \forall x \in \mathfrak{X}$

(c) $\Lambda_0 x = 0 \quad \forall x \in M$.

Proof. (a) $\Rightarrow$ $|\Lambda_0 x| \leq \sum_{i=1}^{N} |\alpha_i| |\Lambda_i x| \leq \max_i |\Lambda_i x| \cdot \sum |\alpha_i| \Rightarrow$ (b).

(b) $\Rightarrow$ If $x \in M$: $|\Lambda_0 x| \leq \gamma \cdot 0 = 0 \Rightarrow$ (c).

"(c) $\Rightarrow$ (a)": Assume $\Lambda_0 x = 0 \ \forall x \in M$. Define $F : \mathfrak{X} \to \mathbb{K}^N$ by $F(x) := (\Lambda_1 x, \dots, \Lambda_N x)$. $\Rightarrow$ F is lin.
If $F(x') = F(x) \Rightarrow \Lambda_i (x' - x) = 0 \ \forall i = 1, \dots, N$
$\Rightarrow x' - x \in M \underset{(c)}{\Rightarrow} \Lambda_0 (x' - x) = 0 \Rightarrow \Lambda_0 x' = \Lambda_0 x$.

Since F is linear, $\tilde{M} := F(\mathfrak{X})$ is a subspace of $\mathbb{K}^N$ and for every $y \in \tilde{M}$ $\exists x_y \in \mathfrak{X}$ s.t. $y = F(x_y)$. We now define $\tilde{f}(y) := \Lambda_0 x_y$, whose the value does not depend on the choice of x_y. Then $\tilde{f}$ is a map $\tilde{M} \to \mathbb{K}$ which is linear and thus also continuous ($\dim \tilde{M} < \infty$). [If $y', y \in \tilde{M} \Rightarrow \exists x', x \in \mathfrak{X}$ s.t. $y' = F(x')$, $y = F(x) \Rightarrow \alpha' y' + \alpha y = F(\alpha' x' + \alpha x)$
$\Rightarrow \tilde{f}(\alpha' y' + \alpha y) = \Lambda_0(\alpha' x' + \alpha x) = \alpha' \tilde{f}(y') + \alpha \tilde{f}(y)$.]
By 12.8, then $\exists \Phi \in (\mathbb{K}^N)^*$ s.t. $\Phi|_{\tilde{M}} = \tilde{f}$
$\Rightarrow \Lambda_0 = \Phi \circ F$ and $\exists \beta \in \mathbb{K}^N$ s.t. $\Phi(y) = \beta \cdot y$. (*)
$\Rightarrow \Lambda_0 x = \sum_{i=1}^{N} \beta_i^* F(x)_i = \sum_{i=1}^{N} \alpha_i \Lambda_i x$ with $\alpha_i = \beta_i^*$. $\square$

(*) If $\mathbb{K} = \mathbb{C}$ this follows from Thrm. 3.1.a). For $\mathbb{K} = \mathbb{R}$, can first use $u(z) = \Phi(\operatorname{Re} z)$, and apply 12.2'. $\Rightarrow \Phi(x) = \operatorname{Re} \beta \cdot x$.

Proof of Thrm 13.1.: Both $\mathbb{R}$ an $\mathbb{C}$ are Hausdorff, and thus by 13.2. $\mathcal{T}' = (\underline{X}'$-weak topology) is Hausdorff for both $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$. $\Rightarrow \{x\}$ closed $\forall x \in \underline{X}$.

For $n \in \mathbb{N}_+$, $r_i > 0$ and $\Lambda_i \in \underline{X}'$, $i = 1, \dots, n$, consider

$$V := \{ x \in \underline{X} \mid |\Lambda_i x| < r_i \ \forall i = 1, \dots, n \}. \qquad (*)$$

Then $V = \bigcap_{i=1}^{n} \Lambda_i^{\leftarrow}(B(0, r_i)) \Rightarrow V \in \mathcal{T}'$ and $0 \in V$. We claim

that $\mathcal{B}_0 := \{ x_0 + V \mid x_0 \in \underline{X},\ V \text{ as above} \}$ is a base for $\mathcal{T}'$. For this, consider $x_0 \in W \in \mathcal{T}'$. $\Rightarrow \exists n \in \mathbb{N}_+, U_i \overset{o}{\subset} \mathbb{K}$, s.t. $x_0 \in \bigcap_{i=1}^{n} \Lambda_i^{\leftarrow} U_i \subset W$. $\Rightarrow \forall i = 1, \dots, n\ \exists r_i > 0$ s.t. $B(\Lambda_i x_0, r_i) \subset U_i$

$\Rightarrow$ if $x \in \overbrace{\bigcap_{i=1}^{n} \Lambda_i^{\leftarrow} B(0, r_i)}^{=:V}$ then $|\Lambda_i(x_0 + x) - \Lambda_i x_0| = |\Lambda_i x| < r_i$

$\Rightarrow \Lambda_i(x_0 + x) \in U_i\ \forall i \Rightarrow x_0 + x \in W$. Thus $x_0 + V \subset W$.

Similarly, we find $x_0 + V = \bigcap_{i=1}^{n} \Lambda_i^{\leftarrow} B(\Lambda_i x_0, r_i) \Rightarrow x_0 \in x_0 + V \in \mathcal{T}'$, and we can conclude that $\mathcal{B}_0$ is a base for $\mathcal{T}'$.

If $0 \le t \le 1$ and $x', x \in V$ (as in $(*)$), then $y = tx + (1-t)x' \in V$ since $|\Lambda_i y| \le t|\Lambda_i x| + (1-t)|\Lambda_i x'| < r_i\ \forall i$. Thus V is convex. If $\alpha \in \mathbb{K}$ with $|\alpha| \le 1$ and $x \in V$, then $\alpha x \in V$ since $|\Lambda_i(\alpha x)| = |\alpha \Lambda_i x| \le |\Lambda_i x| < r_i\ \forall i$. Thus V is also balanced.

Now if $U \in \mathcal{T}'$ and $x_0, y_0 \in \underline{X}$ s.t. $x_0 + y_0 \in U \Rightarrow \exists\, x_0 + y_0 + V \in \mathcal{B}_0$ s.t. $x_0 + y_0 + V \subset U$, V as in $(*)$. Let $V' = \{ x \mid |\Lambda_i x| < \frac{r_i}{2}\ \forall i \}$ when $x', y' \in V' \Rightarrow |\Lambda_i(x' + y')| \le |\Lambda_i x'| + |\Lambda_i y'| < 2 \cdot \frac{r_i}{2} = r_i\ \forall i \Rightarrow x' + y' \in V$.

Therefore, $x_0 \in x_0 + V' \in \mathcal{B}_0$, $y_0 \in y_0 + V' \in \mathcal{B}_0$ and $x_0 + V' + y_0 + V' \subset x_0 + y_0 + V \subset U$. $\therefore$ addition is $\mathcal{T}'$-contin.

Suppose $U \in \mathcal{T}'$, $\alpha \in \mathbb{K}$, $x_0 \in \underline{X}$ and $\alpha x_0 \in U \Rightarrow \exists\, \alpha x_0 + V \in \mathcal{B}_0$ s.t. $\alpha x_0 + V \subset U$ and V as in $(*)$. Set $m := \max_{i=1,\dots,n} \left(\frac{|\Lambda_i x_0|}{r_i} + 1 \right) > 0$

and suppose $\varepsilon > 0$ is so small that $\varepsilon(m + \varepsilon + |\alpha|) < 1$.

Define $V' = \{ x \mid |\Lambda_i x| < \varepsilon r_i\ \forall i \} \Rightarrow x_0 \in x_0 + V' \in \mathcal{B}_0$.

Now if $\beta \in \mathbb{K}$ satisfies $|\beta - \alpha| < \varepsilon$ and $x \in x_0 + V'$, then

$|\Lambda_i(\beta x) - \Lambda_i(\alpha x_0)| = |(\beta - \alpha)\Lambda_i x + \alpha \Lambda_i(x - x_0)|$

$\le |\beta - \alpha||\Lambda_i x| + |\alpha||\Lambda_i(\underbrace{x - x_0}_{})| \le \varepsilon \cdot (|\Lambda_i x_0| + \varepsilon r_i) + |\alpha| \varepsilon r_i$

$< \varepsilon r_i \left(\frac{|\Lambda_i x_0|}{r_i} + \varepsilon + 1 \right) + \overset{\varepsilon V'}{|\alpha| \varepsilon r_i} \le \varepsilon r_i [m + \varepsilon + |\alpha|] < r_i\ \forall i$

$\Rightarrow \beta x - \alpha x_0 \in V \Rightarrow \beta x \in \alpha x_0 + V \in U \therefore \beta(x_0 + V') \subset U$.

Thus scalar multiplication is continuous.
$\therefore$ $\mathcal{T}'$ makes $\mathfrak{X}$ into a topol. vect. space with a local base

$$\mathcal{B}_0 := \{ V \text{ as in } (*) \}.$$

$\Rightarrow$ $\mathfrak{X}$ is a locally convex topol. vect. space.

By def. every $\Lambda \in \mathfrak{X}'$ is now continuous $\Rightarrow$ $\mathfrak{X}' \subset \mathfrak{X}^*$. Suppose then that $\Lambda : \mathfrak{X} \to \mathbb{k}$ is lin. and $\mathcal{T}'$-contin. $\Rightarrow$ $\exists V \in \mathcal{B}_0$ s.t. $V \subset \Lambda^{\leftarrow} B(0,1)$
$\Rightarrow$ $\exists \Lambda_i$, $i = 1, \dots, \tilde{n}$, $\tilde{n} \in \mathbb{N}_+$ and $r_i > 0$ s.t.
If $|\Lambda_i x| < r_i \; \forall i$ then $|\Lambda x| < 1$. Consider $x \in \mathfrak{X}$, and assume that $m := \max_{i=1,\dots,n} |\Lambda_i x| > 0$. Set $\gamma = \max_{1 \le i \le \tilde{n}} \frac{2}{r_i} > 0$

$$\Rightarrow \forall i : \; |\Lambda_i(\tfrac{1}{\gamma m} x)| = \frac{1}{\gamma m} |\Lambda_i x| \le \frac{1}{\gamma} \le \frac{r_i}{2} < r_i$$

$$\Rightarrow |\Lambda x| = \gamma m \, |\Lambda(\tfrac{1}{\gamma m} x)| < \gamma m = \gamma(r) \cdot \max_{i=1,\dots,\tilde{n}} |\Lambda_i x|. \; \forall m = m(x) > 0.$$

If $\Lambda_i = 0 \; \forall i$ $\Rightarrow$ $V = \mathfrak{X}$ $\Rightarrow$ $\Lambda(\mathfrak{X}) \subset B(0,1)$ and since $\Lambda(\mathfrak{X})$ is a subspace $\Rightarrow$ $\Lambda(\mathfrak{X}) = \{0\}$ $\Rightarrow$ $\Lambda = 0 \in \mathfrak{X}'$ (by assumpt.).
Else $\exists j$ s.t. $\Lambda_j \neq 0$ $\Rightarrow$ $\exists y \in \mathfrak{X}$ s.t. $\Lambda_j y \neq 0$.
Consider $x \in \mathfrak{X}$ s.t. $m(x) = 0$, and define $x_n = x + \frac{1}{n} y \in \mathfrak{X}$.
$\Rightarrow$ $|\Lambda_j x_n| = |\Lambda_j x + \frac{1}{n} \Lambda_j y| = \frac{1}{n} |\Lambda_j y| > 0$ $\Rightarrow$ $m(x_n) > 0$
$\Rightarrow$ $|\Lambda x_n| < \gamma \, m(x_n) \; \forall n$. Now $x_n \xrightarrow{n \to \infty} x$ in $\mathcal{T}'$ since for any $\Lambda' \in \mathfrak{X}'$ we have $|\Lambda'(x_n - x)| = \frac{1}{n} |\Lambda' y| \xrightarrow{n \to \infty} 0$
(If $V' \in \mathcal{B}_0$ $\Rightarrow$ $V' = \{x' \mid |\Lambda'_j x'| < r_j \; \forall j = 1, \dots, n'\}$ $\Rightarrow$ $\exists \bar{n} \in \mathbb{N}_+$ s.t. $\forall j = 1, \dots, n' : |\Lambda'_j y| < \bar{n} r'_j$ $\Rightarrow$ $\forall n \ge \bar{n} : x_n - x \in V'$.)
Therefore, by continuity of Λ and Λ_i we have

$$|\Lambda x| = \lim_{n \to \infty} |\Lambda x_n| \le \lim_{n \to \infty} (\gamma \max_{i=1,\dots,\tilde{n}} |\Lambda_i x_n|) = \gamma \cdot m(x) = 0.$$

Thus $\forall x \in \mathfrak{X} : |\Lambda x| \le \gamma \max |\Lambda_i x| \overset{13.3.b)}{\Rightarrow} \exists \alpha_i \in \mathbb{k}$ s.t.

$$\Lambda = \sum_{i=1}^{n} \alpha_i \Lambda_i \in \mathfrak{X}'$$

since $\mathfrak{X}'$ is assumed to be a vector space. $\therefore$ $\mathfrak{X}' = \mathfrak{X}^*$ $\square$

* By 12.7.a), every locally convex t.v.s has a weak topology.
* The proof contains the following result:

A local base for the $\underline{X}'$-weak topology is given by the collection of the sets $V(n, \Lambda, \varepsilon)$, $n \in \mathbb{N}_+$, $\Lambda \in (\underline{X}')^n$, $\varepsilon > 0$, where

$$V(n, \Lambda, \varepsilon) := \{ x \in \underline{X} \mid |\Lambda_i x| < \varepsilon, \forall i = 1, \dots, n \}.$$

<u>13.4. Definition</u>: Suppose $\mathcal{V}$ is a topol. vect. space whose dual <u>$\mathcal{V}^*$ separates points</u> on $\mathcal{V}$. Then the $\mathcal{V}^*$-weak topology is called the <u>weak topology on $\mathcal{V}$.</u>

* We will use the notation τ_W for the weak topology and $\mathcal{V}_W$ to denote $\mathcal{V}$ with its weak topology. Sometimes we will use τ_0 and $\mathcal{V}_0$ to denote the original topology.

* By Theorem 13.1. $\mathcal{V}_W$ is a locally convex topol. vect. space, and $(\mathcal{V}_W)^* = (\mathcal{V}_0)^* = \mathcal{V}^*$. In addition, by definition of $\mathcal{K}$-weak topologies, $\tau_W \subset \tau_0$.

<u>13.5. Proposition</u> Let $\mathcal{V}$ be a topol. vect. space such that $\mathcal{V}^*$ separates points on $\mathcal{V}$. then

(a) If $x_n \in \mathcal{V}$, $n \in \mathbb{N}_+$, then $x_n \to 0$ weakly <u>iff</u> $\Lambda x_n \to 0$ $\forall \Lambda \in \mathcal{V}^*$.

(b) Every originally convergent sequence is weakly convergent.

(c) $E \subset \mathcal{V}$ is weakly bounded <u>iff</u> every $\Lambda \in \mathcal{V}^*$ is bounded on E.

(d) If $\dim \mathcal{V} = \infty$, then every weak neighborhood of 0 contains an infinite dimensional subspace of $\mathcal{V}$.

(e) If $\dim \mathcal{V} = \infty$, $\mathcal{V}_W$ is not normable.

<u>Proof</u>: Exercise. $\square$

13.6. Proposition Assume $\mathcal{V}$ is a locally convex topol. vect. space. If $E \subset \mathcal{V}$ is convex, then the weak closure $\overline{E}^w$ of E is equal to its original closure $\overline{E}$.

Proof: $(\overline{E}^w)^c \in \mathcal{T}_w \subset \mathcal{T}_0$. $\Rightarrow$ $\overline{E} \subset \overline{E}^w$ (= orig. closed).

To get the other direction, assume $x_0 \in \overline{E}^c$. As $\{x_0\}$ is compact and convex and $\overline{E}$ is closed and convex (by 4.11. d), either $E = \emptyset \Rightarrow \overline{E}^w = \emptyset = \overline{E}$ or we can apply Thrm. 12.5. b) $\Rightarrow \exists \Lambda \in \mathcal{V}^*$, $\gamma \in \mathbb{R}$, s.t. $\operatorname{Re}\Lambda x_0 < \gamma < \operatorname{Re}\Lambda x$ $\forall x \in \overline{E}$. Here $\operatorname{Re}\Lambda : \mathcal{X} \to \mathbb{R}$ is a weakly continuous map ($\Lambda \in \mathcal{V}^* = (\mathcal{V}_w)^*$), and thus the set $\{x \mid \operatorname{Re}\Lambda x < \gamma\} = (\operatorname{Re}\Lambda)^{\leftarrow}(-\infty, \gamma)$ is weakly open and contains x_0 but does not intersect $\overline{E} \supset E$. $\Rightarrow x_0 \notin \overline{E}^w$. $\therefore \overline{E}^w = \overline{E}$ $\square$

13.7. Corollaries Let $\mathcal{V}$ be a locally convex t.v.s.

(a) A subspace of $\mathcal{V}$ is originally closed iff it is weakly closed.

(b) If $E \subset \mathcal{V}$ is convex, it is originally dense iff it is weakly dense.

Proof a) Consider a subspace $M \subset \mathcal{V}$. $\Rightarrow M$ convex. By 13.6. $\overline{M} = \overline{M}^w$, and thus $\overline{M} = M \Leftrightarrow \overline{M}^w = M$.

b) By 13.6. $\overline{E} = \overline{E}^w$, and thus $\overline{E} = \mathcal{V} \Leftrightarrow \overline{E}^w = \mathcal{V}$. $\square$

13.8. Proposition Suppose $\mathcal{V}$ is a metrizable locally convex space (e.g. Fréchet). Let $x_n \in \mathcal{V}$, $n \in \mathbb{N}_+$, be such that $x_n \xrightarrow{n\to\infty} x \in \mathcal{V}$ weakly. Then there is a sequence $(y_i)_{i \in \mathbb{N}_+}$ in $\mathcal{V}$ s.t. $y_i \in \operatorname{Hull}\{x_n \mid n \in \mathbb{N}_+\}$ (= convex hull of $\{x_n\}_n$) and $y_i \to x$ in the original topology.

Proof Let $K := \overline{H}^w$, where $H = \operatorname{Hull}\{x_n\}_{n \in \mathbb{N}_+} \overset{\text{Ex.4.5 b)}}{\Rightarrow} H$ is convex. As $x_n \in H$ $\forall n$ $\Rightarrow x \in K$. By 13.6. $K = \overline{H}^w = \overline{H}$ and thus $x \in \overline{H}$. As τ_0 is assumed metrizable, $\exists$ sequence $y_i \in H$, $i \in \mathbb{N}_+$, s.t. $y_i \to x$ originally $\square$

13.9. Definition: (Weak*-topology on $\mathcal{N}^*$)

Assume $\mathcal{N}$ is a topological vector space, and denote $\mathcal{K} := \{ f_x : \mathcal{N}^* \to \mathbb{K} \mid x \in \mathcal{N} \}$ where $f_x(\Lambda) := \Lambda x \;\; \forall x \in \mathcal{N}, \Lambda \in \mathcal{N}^*$. The $\mathcal{K}$-weak topology on $\mathcal{N}^*$ is called the weak*-topology, denoted $\mathcal{T}_{w*}$.

13.10. Proposition: Consider a topol. vect. space $\mathcal{N}$.

(a) $\mathcal{K}$ is a separating vector space of linear functionals on $\mathcal{N}^*$.

(b) The weak*-topology makes $\mathcal{N}^*$ into a locally convex topol. vect. space whose dual is $\mathcal{K}$.

Proof: a) If $x \in \mathcal{N}$, then $f_x(\alpha \Lambda_1 + \beta \Lambda_2) = \alpha \Lambda_1 x + \beta \Lambda_2 x$
$= \alpha f_x(\Lambda_1) + \beta f_x(\Lambda_2) \Rightarrow f_x$ in lin. functional.
Also $f_0(\Lambda) = \Lambda 0 = 0 \;\; \forall \Lambda \in \mathcal{N}^* \Rightarrow f_0 = 0 \in \mathcal{K}$. If $\alpha, \beta \in \mathbb{K}$
$x_1, x_2 \in \mathcal{N} \Rightarrow (\alpha f_{x_1} + \beta f_{x_2})(\Lambda) := \alpha f_{x_1}(\Lambda) + \beta f_{x_2}(\Lambda)$
$= \alpha \Lambda x_1 + \beta \Lambda x_2 = \Lambda(\alpha x_1 + \beta x_2) = f_{\alpha x_1 + \beta x_2}(\Lambda) \;\; \forall \Lambda \in \mathcal{N}^*$.
Since $\alpha x_1 + \beta x_2 \in \mathcal{N} \Rightarrow \alpha f_{x_1} + \beta f_{x_2} \in \mathcal{K}$.
$\therefore \mathcal{K}$ is a vector space. Now if $\Lambda_1, \Lambda_2 \in \mathcal{N}^*$ and $\Lambda_1 \neq \Lambda_2$
$\Rightarrow \exists x \in \mathcal{N}$ s.t. $\Lambda_1 x \neq \Lambda_2 x \Rightarrow f_x(\Lambda_1) \neq f_x(\Lambda_2)$ and $f_x \in \mathcal{K}$. $\therefore \mathcal{K}$ is separating.
b) Follows from a) and theorem 13.1 $\square$

13.11. Theorem (Banach-Alaoglu)

Suppose $\mathcal{N}$ is a topol. vect. space, and $V_0 \in \mathcal{T}_{\mathcal{N}}$ with $0 \in V_0$. Then the set

$$K_{V_0} := \{ \Lambda \in \mathcal{N}^* \mid |\Lambda x| \leq 1 \;\; \forall x \in V_0 \} \quad (= \text{"Polar" of } V_0)$$

is convex, balanced, and weak*-compact.