12. Hahn-Banach Theorems

12.1. Definition. Let $\mathcal{V}$ be a topological vector space. Its dual space is denoted by $\mathcal{V}^*$ (or sometimes $\mathcal{V}'$) and it consists of all continuous linear functionals on $\mathcal{V}$, i.e.,

$$\mathcal{V}^* := \{\Lambda : \mathcal{V} \to \mathbb{K} \mid \Lambda \text{ is contin. and linear}\}.$$

* Clearly, $\Lambda = 0$ belongs to $\mathcal{V}^*$, and thus $\mathcal{V}^* \neq \emptyset$. However, it can happen that $\mathcal{V}^* = \{0\}$ (L^p spaces with $0 < p < 1$). We will see soon that such pathologies do not occur if $\mathcal{V}$ is locally convex. (12.7.a)

* Since the scalar field $\mathbb{K}$ is a topol. vect. space, $\mathcal{V}^*$ is obviously always a vector space. ($\Lambda_1, \Lambda_2 \in \mathcal{V}^* \Rightarrow \alpha\Lambda_1 + \beta\Lambda_2$ is continuous & lin.; $(\alpha\Lambda_1 + \beta\Lambda_2)(x) := \alpha\Lambda_1(x) + \beta\Lambda_2(x)$.)

12.2. Proposition: Suppose X is a complex vector space, and $\Lambda : X \to \mathbb{C}$ is linear. Set $u = \operatorname{Re}\Lambda$. Then u is real-linear and

$$(*) \quad \Lambda x = u(x) - i\,u(ix) \quad \forall x \in X.$$

Conversely, if $u : X \to \mathbb{R}$ is real-linear, then $(*)$ defines a complex-linear map Λ for which $\operatorname{Re}\Lambda = u$, and this map is unique: If $\Lambda' : X \to \mathbb{C}$ is linear and $\operatorname{Re}\Lambda' = u$, then $\Lambda' = \Lambda$.

Proof. Assume Λ linear. For any $z \in \mathbb{C}$, $z = x + iy = x - i(-y) = \operatorname{Re} z - i \operatorname{Re}(iz)$, and thus $\Lambda x = u(x) - i\operatorname{Re}(i\Lambda x) = u(x) - i\,u(ix) \Rightarrow (*)$ holds. & u is obviously real-linear.

Assume $u : X \to \mathbb{R}$ is real-linear, define $\Lambda : X \to \mathbb{C}$ by $(*)$. Then clearly $\Lambda(x+y) = \Lambda x + \Lambda y$. For $r, s \in \mathbb{R}$, $x \in X$,

$\Lambda((r+is)x) = u(rx+six) - iu(rix - sx) = ru(x) + su(ix)$
$- iru(ix) + isu(x) = (r+is)(u(x) - iu(ix)) = (r+is)\Lambda(x)$.
Thus Λ is complex linear, and obviously $\operatorname{Re}\Lambda = u$. If
$\Lambda' : X \to \mathbb{C}$ is linear with $\operatorname{Re}\Lambda' = u \Rightarrow \Lambda' x = u(x) - iu(ix) = \Lambda x$
$\Rightarrow \Lambda' = \Lambda$ $\square$

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12.3. Theorem (Dominated extension)

Let X be a real vector space, and $p: X \to \mathbb{R}$ satisfy

(i) $p(x+y) \le p(x) + p(y)$ $\forall x, y \in X$

and (ii) $p(tx) = t p(x)$ $\forall t \ge 0, x \in X$

Suppose $M \subset X$ is a subspace and $f : M \to \mathbb{R}$ is linear and dominated by p: $f(x) \le p(x)$ $\forall x \in M$.

Then $\exists \Lambda : X \to \mathbb{R}$ s.t. Λ is linear, $\Lambda|_M = f$, and

(*) $-p(-x) \le \Lambda x \le p(x)$ $\forall x \in X$.

(In other words, f has a dominated linear extension.)

Proof: If $M = X$, we can choose $\Lambda = f$, as then
$-\Lambda x = -f(x) = f(-x) \le p(-x)$
$\Rightarrow -p(-x) \le \Lambda x \le p(x)$ $\forall x \in X$.
Assume thus $M \ne X$, and consider $x_1 \in X \setminus M$. Let
$M_1 := \{x + rx_1 \mid x \in M, r \in \mathbb{R}\}$. $\Rightarrow$ M_1 is a vector space. ($x', x \in M$ & $\alpha, \beta, r', r \in \mathbb{R}$ $\Rightarrow \alpha(x' + r'x_1) + \beta(x + rx_1)$
$= \alpha x' + \beta x + (\alpha r' + \beta r)x_1 \in M_1$.) If $x, y \in M$
$\Rightarrow f(x) + f(y) = f(x+y) \le p(x+y) \le p(x - x_1) + p(x_1 + y)$
$\Rightarrow f(x) - p(x - x_1) \le p(y + x_1) - f(y)$. Setting
$\alpha := \sup\{f(x) - p(x - x_1) \mid x \in M\}$ we thus have $\forall x \in M$
$\alpha \le p(x + x_1) - f(x)$ and (by def.) $\alpha \ge f(x) - p(x - x_1)$
$\Rightarrow f(x) - \alpha \le p(x - x_1)$ and $f(x) + \alpha \le p(x + x_1)$.
Therefore, $\forall x \in M$, $t > 0$ it holds that
$t^{-1} f(x) - \alpha = f(t^{-1}x) - \alpha \le p(t^{-1}x - x_1) = t^{-1} p(x - tx_1)$
and $t^{-1} f(x) + \alpha = f(t^{-1}x) + \alpha \le p(t^{-1}x + x_1) = t^{-1} p(x + tx_1)$
Thus if we define $f_1 : M_1 \to \mathbb{R}$ by $f_1(x + rx_1) = f(x) + r\alpha$
we have for $r > 0$: $f_1(x + rx_1) = f(x) + r\alpha \le p(x + rx_1)$,
for $r < 0$: $f_1(x + rx_1) = f(x) - |r|\alpha \le p(x - |r|x_1) = p(x + rx_1)$
and for $r = 0$, by assumption, $f_1(x + rx_1) = f(x) \le p(x + rx_1)$
[NB. The decomposit. is unique: $x - x' = (r' - r)x_1 \in M \Rightarrow r' = r \Rightarrow x' = x$.]

Thus $f_1 \le p|_{M_1}$, $f_1|_M = f$, and f_1 is obviously linear.
$\left(f_1(s'(x'+r'x_1) + s(x+rx_1)) = f_1(s'x'+sx+(s'r'+sr)x_1) \right.$
$\left. = f(s'x'+sx) + (s'r'+sr)\alpha \underset{f \text{ lin.}}{=} s' f_1(x'+r'x_1) + s f_1(x+rx_1) \right)$

To complete the proof, we apply the axiom of choice, or rather, Zorn's lemma. Consider the collection $\mathcal{P} = \{ (M', f') \mid M' \subset X$ subspace, $f': M' \to \mathbb{R}$ linear, s.t. $M \subset M'$, $f'|_M = f$, and $f' \le p|_{M'} \}$.
Define $(M', f') \le (M'', f'')$ if $M' \subset M''$ and $f''|_{M'} = f'$.
Clearly "$\le$" is a partial order in $\mathcal{P}$. (If $(M_i', f_i') \in \mathcal{P}$ for $i = 1,2,3$ $\Rightarrow$ (1) $(M_1', f_1') \le (M_1', f_1')$. (2) $(M_1', f_1') \le (M_2', f_2') \le (M_1', f_1')$
$\Rightarrow M_1' \subset M_2' \subset M_1'$ and $f_2'|_{M_1'} = f_1'$ $\Rightarrow M_1' = M_2'$ and $f_2' = f_1'$.
(3) $(M_1', f_1') \le (M_2', f_2') \le (M_3', f_3')$ $\Rightarrow M_1' \subset M_2' \subset M_3'$
and $f_3'|_{M_1'} = f_2'|_{M_1'} = f_1'$ $\Rightarrow (M_1', f_1') \le (M_3', f_3')$)

Consider a totally ordered non-empty subset $\mathcal{O} \subset \mathcal{P}$. Define
$M_0 := \bigcup_{\exists f' \text{ s.t. } (M', f') \in \mathcal{O}} M'$ $\Rightarrow 0 \in M_0$, and if $r', r \in \mathbb{R}$ and $x, y \in M_0$
$\Rightarrow \exists M_1', M_2'$ s.t. $x \in M_1'$, $y \in M_2'$ and either $M_1' \subset M_2'$ or $M_2' \subset M_1'$ $\Rightarrow \exists (M', f') \in \mathcal{O}$ s.t. $x, y \in M'$ and, as M' is subspace, $r'x + ry \in M' \subset M_0$. Thus M_0 is a subspace.
Now for $x \in M_0$ $\Rightarrow \exists (M', f') \in \mathcal{O}$ s.t. $x \in M'$, and set $f_0(x) := f'(x) \le p(x)$. $f_0(x)$ is indep. of choice of $(M', f') \in \mathcal{O}$; since if $(M'', f'') \in \mathcal{O}$ with $x \in M''$ $\Rightarrow$ either $(M'', f'') \le (M', f')$, when $f'|_{M''} = f''$ implies $f'(x) = f''(x)$, or $(M', f') \le (M'', f'')$, when $f''|_{M'} = f'$ implies $f''(x) = f'(x)$. In addition, if $x, y \in M_0$
$\Rightarrow \exists (M', f') \in \mathcal{O}$ s.t. $x, y \in M'$ $\Rightarrow r'x + ry \in M'$ $\Rightarrow f_0(r'x + ry)$
$= f'(r'x+ry) \underset{f' \text{ lin.}}{=} r' f'(x) + r f'(y) = r' f_0(x) + r f_0(y)$.

Obviously, also $x \in M \Rightarrow \exists (M', f') \in \mathcal{O}$ s.t. $x \in M' \subset M_0$ and $f_0(x) = f'(x) = f(x)$.
Thus $(M_0, f_0) \in \mathcal{P}$ and $(M', f') \le (M_0, f_0)$ $\forall (M', f') \in \mathcal{O}$. This proves that $\mathcal{O}$ has an upper bound in $\mathcal{P}$.

Therefore, we can apply Zorn's lemma, and conclude that there is a maximal $(\check{M}, \Lambda) \in \mathcal{P}$. If $\check{M} \ne X$, we can apply the first part of the proof, and construct $(\bar{M}_1, \bar{f}_1)$ s.t. $(\bar{M}, \bar{f}) < (\bar{M}_1, \bar{f}_1)$. ↯ Thus $\check{M} = X$ and $\Lambda : X \to \mathbb{R}$ is linear, $\Lambda|_M = f$ and $\Lambda \le p$. As in the beginning of the proof $\Rightarrow$ $(*)$ holds as well. $\square$

12.4. Corollaries

(a) Let X be a vector space, and assume $p: X \to \mathbb{R}$ is a <u>seminorm</u>. If $M \subset X$ is a subspace and $f: M \to \mathbb{K}$ is linear, with $|f(x)| \leq p(x)$ $\forall x \in M$, then $\exists \Lambda: X \to \mathbb{K}$ s.t. Λ is linear, $\Lambda|_M = f$, and $|\Lambda(x)| \leq p(x)$ $\forall x$.

(b) If X is a normed space and $x_0 \in X$, there is $\Lambda \in X^*$ such that $\Lambda x_0 = \|x_0\|$ and $|\Lambda x| \leq \|x\|$ $\forall x \in X$.

Proof. a) If $\mathbb{K} = \mathbb{R}$, we can directly apply Thrm. 12.3. ($\Rightarrow -p(x) = -p(-x) \leq \Lambda x \leq px$). Else $\mathbb{K} = \mathbb{C}$. Define $u = \mathrm{Re}\, f \Rightarrow u \leq |u| \leq |f| \leq p$ on M and u is real-linear. By Thrm. 12.3. $\exists v: X \to \mathbb{R}$ s.t. $v \leq p$, v is real-linear and $v|_M = u$.
By Proposition 12.2., then $\Lambda x := v(x) - iv(ix)$ defines a complex-linear map $X \to \mathbb{C}$ and for $x \in M$: $\Lambda x = u(x) - iu(ix) = f(x)$. In addition, $\forall x \in X$ $\exists \alpha \in \mathbb{C}$ s.t. $|\alpha| = 1$ and $|\Lambda x| = \alpha \Lambda x$ (If $\Lambda x \neq 0$, choose $\alpha = \frac{(\Lambda x)^*}{|\Lambda x|}$, else $\alpha = 1$.)
$\Rightarrow |\Lambda x| = \mathrm{Re}\, |\Lambda x| = \mathrm{Re}(\alpha \Lambda x) = \mathrm{Re}\, \Lambda(\alpha x) = v(\alpha x)$
$\leq p(\alpha x) = p(x)$ (p = seminorm.). ■

b) Let $M = \mathrm{span}(x_0)$ and define $f(\alpha x_0) \overset{(*)}{=} \alpha \|x_0\|$ $\forall \alpha \in \mathbb{K}$. Then $f: M \to \mathbb{K}$ is linear and $|f(\alpha x_0)| = |\alpha| \|x_0\| = \|\alpha x_0\| \Rightarrow |f| \leq p(x)$ with $p(x) := \|x\|$ for $x \in X$. As p is a (semi)norm $\underset{a)}{\Rightarrow}$ $\exists \Lambda: X \to \mathbb{K}$ s.t. Λ is linear, $\Lambda(x_0) = \|x_0\|$, and $\forall x \in X$: $|\Lambda x| \leq p(x) = \|x\|$. $\Rightarrow \Lambda$ is bounded $\underset{\text{Ex. 5.1.}}{\Rightarrow}$ Λ is continuous. $\therefore \Lambda \in X^*$. □

(*) If $x_0 = 0$, $M = \{0\}$ and the def. reads $f(0) = 0$. Else, $x \in M \Rightarrow \exists! \alpha \in \mathbb{K}$ s.t. $x = \alpha x_0$.

Ⓐ Appendix to p. 87.

12.2.' Corollary. Let $\mathcal{V}$ be a complex topol. vect. space. Suppose $\Lambda: \mathcal{V} \to \mathbb{C}$ is lin. and denote $u = \operatorname{Re}\Lambda$. Then $\Lambda \in \mathcal{V}^*$ iff u is contin. In addition, if $u: \mathcal{V} \to \mathbb{R}$ is real-linear and contin., then $\exists! \Lambda \in \mathcal{V}^*$ s.t. $u = \operatorname{Re}\Lambda$.

Proof: As $z \mapsto \operatorname{Re} z$ is continuous map $\mathbb{C} \to \mathbb{R}$, $\Lambda \in \mathcal{V}^* \Rightarrow u = \operatorname{Re}\Lambda$ is contin. If $\Lambda: \mathcal{V} \to \mathbb{C}$ is lin. and $u = \operatorname{Re}\Lambda$, then $(*)$ holds in Prop. 12.2. As $\mathcal{V}$ is a complex t.v.s., the map $x \mapsto ix$ is contin, and thus u continuous $+ (*) \Rightarrow \Lambda$ is continuous.
For the second statement, assume $u: \mathcal{V} \to \mathbb{R}$ is $\mathbb{R}$-lin. and cont. By 12.2, the formula $(*)$ defines $\Lambda: \mathcal{V} \to \mathbb{C}$ which is lin. and (as above) continuous. In addition, $\Lambda' \in \mathcal{V}^*$ and $\operatorname{Re}\Lambda' = u \Rightarrow \Lambda' = \Lambda$. $\square$

12.5. Theorem (Separation by functionals)

Let $\mathcal{V}$ be a topol. vect. space, and assume $A, B \subset \mathcal{V}$ are nonempty, convex and disjoint.

(a) If A is open, there is $\Lambda \in \mathcal{V}^*$ and $\gamma \in \mathbb{R}$ s.t.

$$\operatorname{Re}\Lambda x < \gamma \leq \operatorname{Re}\Lambda y \quad \forall x \in A, y \in B.$$

(b) If A is compact, B is closed, and $\mathcal{V}$ is locally convex, then $\exists \Lambda \in \mathcal{V}^*$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ s.t.

$$\operatorname{Re}\Lambda x < \gamma_1 < \gamma_2 < \operatorname{Re}\Lambda y \quad \forall x \in A, y \in B$$

Proof: We first prove the theorem assuming $\mathbb{K} = \mathbb{R}$. Then of course $\operatorname{Re}\Lambda = \Lambda$.
"(a)" Choose $a_0 \in A$, $b_0 \in B$, and set $x_0 := b_0 - a_0$, $C := A - B + x_0$. Then $0 \in C$ and, as $C = \bigcup_{b \in B} (x_0 - b + A)$ and A is open $\Rightarrow C$ is open.

If $0 \le t \le 1$ and $a', a \in A$, $b', b \in B$, then
$t(a-b+x_0)+(1-t)(a'-b'+x_0)$
$= ta + (1-t)a' - (tb+(1-t)b') + x_0 \in C$. This shows that C is convex. As it is a neighborhood of 0, it is also absorbing (Ex. 6.2.) and we can consider its Minkowski functional $\mu_C : V \to \mathbb{R}$. By Ex. 6.5 a) & b), μ_C satisfies assumptions (i) & (ii) in Thrm 12.3. Now $x_0 \notin C$, as else $\exists a \in A, b \in B$ s.t. $a = b \Rightarrow A \cap B \neq \emptyset$. ↯. Therefore, by Ex. 6.5. d) we have $\mu_C(x_0) \ge 1$. As above, $A \cap B = \emptyset \Rightarrow x_0 \neq 0$, and thus $M := \operatorname{span}(x_0)$ is a 1-dim. subspace of V. Define $f : M \to \mathbb{R}$ by $f(rx_0) = r \ \forall r \in \mathbb{R}$ which is obviously a linear map. In addition, for $t \ge 0$, we have

$$f(-tx_0) = -t \le 0 \le \mu_C(tx_0) \quad \text{and} \quad f(tx_0) = t \le t\mu_C(x_0) \overset{\text{Ex. 6.5. b)}}{=} \mu_C(tx_0)$$

Therefore, $f \le \mu_C|_M$ and Theorem 12.3. shows that $\exists \Lambda : V \to \mathbb{R}$ s.t. Λ is linear, $\Lambda|_M = f$ and $-\mu_C(-x) \le \Lambda x \le \mu_C(x) \ \forall x \in V$.
If $x \in C \underset{\text{Ex. 5.5. d)}}{\Rightarrow} \mu_C(x) \le 1 \Rightarrow \Lambda x \le 1$ and $\Lambda(-x) \ge -\mu_C(x) \ge -1$.

Thus $|\Lambda| \le 1$ on $C \cap (-C)$ which is a neighb. of 0. By Thrm. 5.3, Λ is continuous $\Rightarrow \Lambda \in V^*$.

Since C is open, we have $\mu_C(x) < 1 \ \forall x \in C$. To see this note that $x_n := (1+\frac{1}{n})x \to x$, since V is a topol. vect. space. If $x_n \notin C \ \forall n \Rightarrow x_n \in C^c \ \forall n \Rightarrow x \in \overline{C^c} = C^c$ ↯. Thus $\exists n$ s.t. $x_n \in C \Rightarrow \frac{n+1}{n} x \in C \Rightarrow \mu_C(x) \le \frac{n}{n+1} < 1$. Therefore, for any $a \in A, b \in B$, we have $\Lambda a - \Lambda b + 1 = \Lambda a - \Lambda b + f(x_0)$
$= \Lambda(a-b+x_0) \le \mu_C(a-b+x_0) < 1$, as $a-b+x_0 \in C$.
$\Rightarrow \Lambda a < \Lambda b$. In particular, Λ is not constant, and thus by Lemma 12.6. below, Λ is an open mapping.
$\Rightarrow \Lambda(A)$ is open $\Rightarrow \gamma := \sup \Lambda(A) \notin \Lambda(A) \Rightarrow \Lambda a < \gamma \le \Lambda b$ ■

"b)" By Thrm. 4.7 and, since V is now locally convex, $\exists V \in \mathcal{T}_V$ s.t. $0 \in V$, V is convex, and $(A+V) \cap B = \emptyset$.
Since then $A' = A + V$ is convex and open we can apply a) to it
$\Rightarrow \exists \Lambda \in V^*$ and $\gamma \in \mathbb{R}$ s.t. $\Lambda a' < \gamma \le \Lambda b \quad \forall a' \in A', b \in B$.
Since Λ is contin., $\Lambda(A)$ is compact subset of $\mathbb{R} \Rightarrow$
$m_A := \sup \Lambda(A) \in \Lambda(A) \underset{0 \in V}{\subset} \Lambda(A+V) \Rightarrow \exists \varepsilon > 0$ s.t. $B(m_A, \varepsilon) \subset \Lambda(A+V)$.

Thus for instance with $\gamma_1 = m_A + \frac{\varepsilon}{3}$, $\gamma_2 = m_A + \frac{2}{3}\varepsilon$

$\Rightarrow \Lambda a \le m_A < \gamma_1 < \gamma_2 < m_A + \varepsilon \le \gamma \le \Lambda b \quad \forall a \in A, b \in B$. ■

12.6. Lemma Let $\mathcal{V}$ be a topol. vect. space. If $\Lambda \in \mathcal{V}^*$ and $\Lambda \neq$ constant, then Λ is an open mapping.

Proof. Now $\exists x_0 \in \mathcal{V}$ s.t. $\Lambda x_0 \neq \Lambda 0 = 0$. Consider $V \in \mathcal{T}_{\mathcal{V}}$, and $x \in V$. By Thrm. 4.12. $\exists U_0 \in \mathcal{T}_{\mathcal{V}}$ s.t. $0 \in U_0 \subset V - x$ and U_0 is balanced. By 4.14 a), $\exists n_0 \in \mathbb{N}_+$ s.t. $x_0 \in n_0 U_0$.
Then, if $y \in B(\Lambda x, \frac{|\Lambda x_0|}{n_0}) \Rightarrow \exists \alpha \in \mathbb{K}$ with $|\alpha| \leq 1$, s.t.

$$y = \Lambda x + \alpha \frac{\Lambda x_0}{n_0}$$
$$= \Lambda(x + \frac{\alpha}{n_0} x_0) \in \Lambda(x + \alpha U_0) \subset \Lambda(x + U_0) \subset \Lambda(V).$$

$\therefore B(\Lambda x, \underbrace{n_0^{-1}|\Lambda x_0|}_{>0}) \subset \Lambda(V)$. Thus $\Lambda(V)$ is open. □

To complete the proof of 12.5. we need to consider the case $\mathbb{K} = \mathbb{C}$. However, as $\mathcal{V}$ is then also a real vector space, we can apply the previous result and conclude that there is a real-linear $u: \mathcal{V} \to \mathbb{R}$ which is continuous and for which the desired inequality holds. However, by Corollary 12.2.' then $\exists! \Lambda \in \mathcal{V}^*$ s.t. $u = \mathrm{Re}\,\Lambda$, and thus this Λ has all the required properties. □

12.7. Corollaries:

Consider a locally convex topol. vect. space $\mathcal{V}$.

(a) $\mathcal{V}^*$ separates points on $\mathcal{V}$:
For any $x_1, x_2 \in \mathcal{V}$, $x_1 \neq x_2$, there is $\Lambda \in \mathcal{V}^*$ s.t. $\Lambda(x_1) \neq \Lambda(x_2)$.

(b) If $M \subset \mathcal{V}$ is a subspace and $x_0 \in \mathcal{V} \setminus \overline{M}$, then $\exists \Lambda \in \mathcal{V}^*$ s.t. $\Lambda x_0 = 1$ but $\Lambda x = 0 \ \forall x \in M$.

(c) →

Proof: "a)" Apply 12.5.b) to $A = \{x_1\}$, $B = \{x_2\}$ ($\Rightarrow A \cap B = \emptyset$.)
"b)" Set $A = \{x_0\}$ and $B = \overline{M}$. Then $A \cap B = \emptyset$, $\overline{M}$ is a subspace $\Rightarrow$ convex, and A is compact. Thus by 12.5.b) $\Rightarrow \exists \Lambda \in \mathcal{V}^*$ s.t. $\Lambda(A) = \{\Lambda x_0\}$ and $\Lambda(\overline{M})$ are disjoint $\Rightarrow \Lambda x_0 \notin \Lambda(M) =$ subspace of $\mathbb{K}$.
$\Rightarrow \Lambda(M) = \{0\}$ and $\Lambda x_0 \neq 0$. Therefore $\frac{1}{|\Lambda x_0|}\Lambda \in \mathcal{V}^*$ has the required properties. □