10.11. Theorem Consider a collection Γ of continuous and linear maps $\mathcal{V}_1 \to \mathcal{V}_2$, where $\mathcal{V}_1, \mathcal{V}_2$ are topol. vect. spaces. Assume $K \subset \mathcal{V}_1$ is compact and convex, and that every orbit $\Gamma(x)$ with $x \in K$ is bounded in $\mathcal{V}_2$.

Then $\exists B \subset \mathcal{V}_2$ which is bounded, and $\Lambda(K) \subset B \; \forall \Lambda \in \Gamma$.

Proof: Define $B := \bigcup_{x \in K} \Gamma(x)$. Consider $0 \in W_2 \in \mathcal{T}_{\mathcal{V}_2}$.

$\Rightarrow \exists U, W \in \mathcal{T}_{\mathcal{V}_2}$ s.t. $0 \in U$, $0 \in W$, U and W are balanced, and $\bar{U} + \bar{U} \subset W \subset W_2$. Set $E := \bigcap_{\Lambda \in \Gamma} \Lambda^{\leftarrow} \bar{U}$, which is a closed set in $\mathcal{V}_1$, as each $\Lambda \in \Gamma$ is continuous. If $x \in K$ $\Rightarrow \Gamma(x)$ is bounded $\Rightarrow \Gamma(x) \subset nU$ for some $n \in \mathbb{N}_+$.
$\Rightarrow \Lambda x \in nU \subset n\bar{U} \; \forall \Lambda \in \Gamma \Rightarrow \Lambda(\frac{1}{n}x) \in \bar{U} \; \forall \Lambda \in \Gamma$
$\Rightarrow \frac{1}{n} x \in E \Rightarrow x \in nE$. Therefore,
$$K = \bigcup_{n=1}^{\infty} (K \cap (nE)).$$ By 10.4. K is of 2nd cat. in K, and thus $\exists n \in \mathbb{N}_+$ s.t. $K \cap (nE)$ has a non-empty interior in K $\Rightarrow \exists n \in \mathbb{N}_+$, $x_0 \in K$ and $V \in \mathcal{T}_{\mathcal{V}_1}$ s.t. $0 \in \underbrace{V}_{V \text{ is balanced}}$ and $K \cap (x_0 + V) \subset nE$.

As $K - x_0$ is compact, it is bounded, and thus $\exists p > 1$ s.t. $K - x_0 \subset pV \Rightarrow K \subset x_0 + pV$.

Consider then an arbitrary $x \in K$, and let $z = (1 - \frac{1}{p}) x_0 + \frac{1}{p} x$. As $0 < \frac{1}{p} < 1$ and K is convex $\Rightarrow z \in K$. But also $z - x_0 = \frac{1}{p}(x - x_0) \in V$, and thus $z \in K \cap (x_0 + V) \subset nE$. $\Rightarrow \Lambda z \in n\Lambda E \subset n\bar{U} \; \forall \Lambda \in \Gamma$
$\Rightarrow \Lambda x = \Lambda(pz - (p-1)x_0) \in pn\bar{U} - (p-1)n\bar{U} = pn(\bar{U} - \frac{p-1}{p}\bar{U})$
$\underset{p}{\subset} pn(\bar{U} + \bar{U}) \subset pnW$. $\forall \Lambda \in \Gamma$. Thus $y \in B \Rightarrow \exists x \in K, \Lambda \in \Gamma$
U bal. $\Rightarrow \bar{U}$ bal.
s.t. $y = \Lambda x \Rightarrow y \in pnW$, and thus $B \subset pnW \subset^{*} t \frac{pn}{t} W \subset tW \subset tW_2$ for all $t > pn$. $\therefore$ B is bounded $\square$

* Convexity is required here.

11. Open mappings and closed graphs

11.1. Theorem (Open mapping theorem)

Suppose $\Lambda : \mathcal{F}_1 \to \mathcal{N}_2$ is continuous and linear, where $\mathcal{F}_1$ is an F-space, $\mathcal{N}_2$ is a topol. vect. space. If $\Lambda(\mathcal{F}_1)$ is of the 2nd category in $\mathcal{N}_2$, then

(a) $\Lambda(\mathcal{F}_1) = \mathcal{N}_2$
(b) Λ is an open mapping
(c) $\mathcal{N}_2$ is an F-space.

Proof. Here b) $\Rightarrow$ a): If Λ is open, then $\Lambda(\mathcal{F}_1)$ is an open subspace (5.1.b) of $\mathcal{N}_2 \Rightarrow \exists V \in \mathcal{T}_{\mathcal{N}_2}$ s.t. $0 \in V \subset \Lambda(\mathcal{F}_1) \underset{4.14.a)}{\Rightarrow} \mathcal{N}_2 = \bigcup_{n=1}^{\infty} (nV) \underset{\Lambda(\mathcal{F}_1)\text{ subspace}}{\subset} \Lambda(\mathcal{F}_1) \subset \mathcal{N}_2$.
$\Rightarrow$ a).

Let us first prove b). Let d be an invariant compatible metric on $\mathcal{F}_1$, and consider $0 \in V_0 \in \mathcal{T}_{\mathcal{F}_1}$ $\Rightarrow \exists r > 0$ s.t. $B(0,r) \subset V_0$. Define $W_n = B(0, 2^{-n}r)$, $n \in \mathbb{N}_0$, $\Rightarrow W_{n+1} \subset W_n \ \forall n \geq 0$, $W_0 \subset V_0$. We claim that $\forall N \in \mathbb{N}_+$ $\exists U \in \mathcal{T}_{\mathcal{N}_2}$, s.t. $0 \in U$ and $U \subset \overline{\Lambda(W_N)}$. Now $x, y \in W_{N+1}$ $\Rightarrow d(x-y, 0) = d(x,y) \leq d(x,0) + d(0,y) < 2^{-(N+1)}(r+r) = 2^{-N}r$.
Thus $W_{N+1} - W_{N+1} \subset W_N \Rightarrow \Lambda W_{N+1} - \Lambda W_{N+1} = \Lambda(W_{N+1} - W_{N+1}) \subset \Lambda W_N$
$\underset{4.11.b)}{\Rightarrow} \overline{\Lambda W_{N+1}} - \overline{\Lambda W_{N+1}} = \overline{\Lambda W_{N+1}} + \overline{(-\Lambda W_{N+1})} \subset \overline{\Lambda W_{N+1} - \Lambda W_{N+1}} \subset \overline{\Lambda W_N}$.
As $0 \in W_{N+1} \in \mathcal{T}_{\mathcal{F}_1} \overset{4.14.a)}{\Rightarrow} \mathcal{F}_1 = \bigcup_{n \in \mathbb{N}_+} (nW_{N+1}) \Rightarrow \Lambda(\mathcal{F}_1) = \bigcup_{n=1}^{\infty} (n\Lambda(W_{N+1}))$

Since $\Lambda(\mathcal{F}_1)$ is of 2nd cat. $\Rightarrow \exists n \in \mathbb{N}_+$ s.t. $n\Lambda(W_{N+1})$ is of 2nd cat. $\Rightarrow \Lambda(W_{N+1}) = \frac{1}{n}(n\Lambda(W_{N+1}))$ is of 2nd. cat. in $\mathcal{N}_2$ ($x \mapsto \frac{1}{n}x$ is homeo.) $\Rightarrow \exists x_0 \in \overline{\Lambda(W_{N+1})}$ and $U \in \mathcal{T}_{\mathcal{N}_2}$ s.t. $0 \in U$ and $x_0 + U \subset \overline{\Lambda(W_{N+1})} \Rightarrow U = x_0 + U - x_0 \subset \overline{\Lambda(W_{N+1})} - x_0$ $\subset \overline{\Lambda(W_{N+1})} - \overline{\Lambda(W_{N+1})} \subset \overline{\Lambda W_N}$. Thus the claim holds for this U.

Consider then $y_1 \in \overline{\Lambda W_1}$. By the previous result, for any $N \in \mathbb{N}_+$, $\exists U_N \in \mathcal{T}_{\mathcal{N}_2}$ s.t. $0 \in U_N \subset \overline{\Lambda(W_{N+1})}$. Thus if $y \in \overline{\Lambda W_N} \Rightarrow \emptyset \neq \Lambda W_N \cap (y - U_N) \subset \Lambda W_N \cap (y - \overline{\Lambda W_{N+1}})$. Therefore, we can iteratively choose a sequence $(y_n)_{n \in \mathbb{N}_+}$ as follows: Suppose $y_n \in \overline{\Lambda W_n}$ is given. Then $\exists x_n \in W_n$ s.t. $\Lambda x_n \in y_n - \overline{\Lambda W_{n+1}}$. Set $y_{n+1} := y_n - \Lambda x_n$.

$\Rightarrow y_{n+1} \in \overline{\Lambda W_{n+1}}$ and we can iterate further.
Since then $x_n \in W_n \Rightarrow d(x_n, 0) < 2^{-n} r$. $\forall n \in \mathbb{N}_+$.
Define $z_n = \sum_{k=1}^{n} x_k \Rightarrow d(z_{n+m}, z_n) = d(z_{n+m} - z_n, 0)$

$$= d\left(\sum_{k=n+1}^{n+m} x_k, 0\right) \leq \sum_{k=n+1}^{n+m} d(x_k, 0) \leq \sum_{k=n+1}^{n+m} 2^{-k} r < 2^{-n} r.$$

Thus $(z_n)_{n \in \mathbb{N}_+}$ is Cauchy in $\mathcal{F}_1 \Rightarrow \exists z = \lim_{n\to\infty} z_n$
and $d(z, 0) = \lim_{n\to\infty} d(z_n, 0) \leq \sum_{k=1}^{\infty} d(x_k, 0)$

$$< \sum_{k=1}^{\infty} 2^{-k} r = 2^{-1} \cdot \frac{1}{1-\frac{1}{2}} r = r \Rightarrow z \in W_0 \subset V_0.$$

In addition, by continuity of $\Lambda \Rightarrow$

$$\Lambda z = \lim_{n\to\infty} \sum_{k=1}^{n} \Lambda x_k, \text{ where } \sum_{k=1}^{n} \Lambda x_k = \sum_{k=1}^{n} (y_k - y_{k+1})$$

$= y_1 - y_{n+1}$. We claim that $y_n \xrightarrow{n\to\infty} 0$. Then
$\Lambda z = y_1 \Rightarrow y_1 \in \Lambda(V_0)$, and we have proven
$\overline{\Lambda W_1} \subset \Lambda(V_0)$. As $\overline{\Lambda W_1}$ contains $U_1 \in \mathcal{T}_{\mathcal{N}_2}$ s.t. $0 \in U_1$
$\Rightarrow 0 \in U_1 \subset \Lambda(V_0)$. Therefore, if $x \in V \in \mathcal{T}_{\mathcal{F}_1} \Rightarrow$
$0 \in V - x \in \mathcal{T}_{\mathcal{F}_1} \Rightarrow \exists U_x \in \mathcal{T}_{\mathcal{N}_2}$ s.t. $0 \in U_x$ and $U_x \subset \Lambda(V-x)$
$= \Lambda V - \Lambda x \Rightarrow \Lambda x + U_x \subset \Lambda V$. Therefore,
$\Lambda V = \bigcup_{x \in V} (\Lambda x + U_x)$ is open in $\mathcal{N}_2$, which proves that
Λ is open mapping.

To prove $y_n \to 0$, consider $0 \in U \in \mathcal{T}_{\mathcal{N}_2}$. $\Rightarrow$
$\exists$ balanced $V \in \mathcal{T}_{\mathcal{N}_2}$ s.t. $0 \in V$, $V + V \subset U$. Since $y_n \in \overline{\Lambda W_n}$
$\Rightarrow \emptyset \neq \Lambda W_n \cap (y_n + V) \Rightarrow \exists x_n' \in W_n$ s.t. $\Lambda x_n' \in y_n + V$.
But then $d(x_n', 0) < 2^{-n} r \xrightarrow{n\to\infty} 0$, $\Rightarrow x_n' \to 0$ in $\mathcal{F}_1$
$\underset{\Lambda \text{ contin.}}{\Rightarrow} \Lambda x_n' \to 0$ in $\mathcal{N}_2 \Rightarrow \exists N \in \mathbb{N}_+$ s.t. $\Lambda x_n' \in V \; \forall n \geq N$.
$\Rightarrow y_n = y_n - \Lambda x_n' + \Lambda x_n' \in -V + V = V + V \subset U$.
i.e. $y_n \to 0$ in $\mathcal{N}_2$. This completes proof of b), $\Rightarrow$ a) holds.
Let $N := \ker \Lambda \underset{\Lambda \text{ contin.}}{\Rightarrow} N$ is closed in $\mathcal{F}_1$. By

Exercise 5.4. $\exists! \, \Phi : \mathcal{F}_1/N \to \mathcal{N}_2$ s.t. $\Lambda = \Phi \circ \pi_N$
and this Φ is linear, continuous and open (by b)!).
By a), $\Phi(\mathcal{F}_1/N) = \Lambda(\mathcal{F}_1) = \mathcal{N}_2 \Rightarrow \Phi$ is onto.
In addition, if $\Phi(x+N) = \Phi(x'+N) \Rightarrow \Lambda(x) = \Lambda(x')$
$\Rightarrow \Lambda(x - x') = 0 \Rightarrow x - x' \in \ker \Lambda = N \Rightarrow x \sim x'$ i.e. $x + N = x' + N$.
Thus Φ is bijective $\Rightarrow \Phi^{-1}$ linear and continuous
$\Rightarrow \Phi$ homeo. Since by 3.3. f) $\mathcal{F}_1/N$ is an F-space $\Rightarrow \mathcal{N}_2 =$ F-space $\square$ see p. 85

11.2. Corollaries:

(a) If $\mathfrak{X}_1, \mathfrak{X}_2$ are F-spaces and $\Lambda: \mathfrak{X}_1 \to \mathfrak{X}_2$ is continuous and linear and onto, then Λ is open. If Λ is also one-to-one, then $\Lambda^{-1}: \mathfrak{X}_2 \to \mathfrak{X}_1$ is continuous and linear.

(b) If $\mathcal{B}_1, \mathcal{B}_2$ are Banach spaces, and $\Lambda: \mathcal{B}_1 \to \mathcal{B}_2$ is bijective, continuous and linear, then $\exists\, m, M > 0$ such that

$$(*) \quad m\|x\| \leq \|\Lambda x\| \leq M\|x\| \quad \forall x \in \mathcal{B}_1.$$

(c) If $\mathcal{T}_1 \subset \mathcal{T}_2$, and $\mathcal{T}_1, \mathcal{T}_2$ are both topologies on a vector space $\mathfrak{X}$ s.t. both $(\mathfrak{X}, \mathcal{T}_1)$ and $(\mathfrak{X}, \mathcal{T}_2)$ are F-spaces, then $\mathcal{T}_1 = \mathcal{T}_2$.

(Thus F-space topologies cannot be refined into new F-space topologies.)

Proof. a) $\Lambda(\mathfrak{X}_1) = \mathfrak{X}_2 \overset{10.4.}{\Longrightarrow} \Lambda(\mathfrak{X}_1)$ 2nd cat. in $\mathfrak{X}_2$ $\underset{11.1.}{\Longrightarrow}$ Λ is open. If Λ is also 1-1 $\Rightarrow \exists \Lambda^{-1}$ and Λ^{-1} is continuous and linear.

b) By a), $\Lambda^{-1}: \mathcal{B}_2 \to \mathcal{B}_1$ is contin. and linear $\Rightarrow \exists\, m^{-1} > 0$ s.t. $\forall y \in \mathcal{B}_2: \|\Lambda^{-1}y\| \leq m^{-1}\|y\|$. Thus for $y = \Lambda x \Rightarrow \|x\| \leq m^{-1}\|\Lambda x\|$ $\Rightarrow \|\Lambda x\| \geq m\|x\|$. We applied here $\tilde{\Lambda}$ cont. lin. $\overset{8.3.}{\Longrightarrow} \tilde{\Lambda}$ bounded $\overset{Ex.5.1.}{\Longrightarrow} \|\tilde{\Lambda}\| < \infty \Rightarrow \forall x \neq 0: \|\tilde{\Lambda} x\| = \|x\| \, \|\tilde{\Lambda}\frac{x}{\|x\|}\| \leq \|x\| \, \|\tilde{\Lambda}\| \Rightarrow \exists M > 0$ s.t. $\forall x: \|\tilde{\Lambda} x\| \leq M\|x\|$. Since $\Lambda: \mathcal{B}_1 \to \mathcal{B}_2$ is also cont. lin. we have proven $(*)$.

c) Let $V_1 = (\mathfrak{X}, \mathcal{T}_1)$ and $V_2 = (\mathfrak{X}, \mathcal{T}_2)$. Then $id: \mathfrak{X} \to \mathfrak{X}$ is bijective and linear. Also if $V \in \mathcal{T}_1 \Rightarrow id^{\leftarrow}(V) = V \in \mathcal{T}_2$, and thus id is continuous as a map $V_2 \to V_1$. Since both V_1, V_2 are F-spaces, by a) $\Rightarrow id^{-1} = id: V_1 \to V_2$ is continuous and thus $V \in \mathcal{T}_2 \Rightarrow V = id^{\leftarrow}(V) \in \mathcal{T}_1$. $\therefore \mathcal{T}_1 = \mathcal{T}_2$ $\square$

11.3. Definition The graph of $f : X \to Y$ is denoted by $\mathcal{G}(f) := \{(x, f(x)) \mid x \in X\} \subset X \times Y$.

* Typically a graph of a continuous function is closed:

11.4. Proposition: Consider $f : X \to Y$, where X, Y have topologies $\mathcal{T}_X$, $\mathcal{T}_Y$, and $\mathcal{T}_Y$ is Hausdorff. If f is continuous then the graph of f is a closed subset of $X \times Y$ (in the product topology).

Proof: Set $G := \{(x, f(x)) \mid x \in X\}$, and $\Omega := G^c$. Consider $(x_0, y_0) \in \Omega \Rightarrow y_0 \neq f(x_0)$. As Y is Hausdorff, $\exists U, V \in \mathcal{T}_Y$ s.t. $f(x_0) \in U$ and $y_0 \in V$ but $U \cap V = \phi$. Since f is contin. $\Rightarrow W = f^{\leftarrow} U \in \mathcal{T}_X$ $\Rightarrow x_0 \in W$ and $f(W) \subset U \Rightarrow \forall x \in W,\ f(x) \notin V$, and thus $W \times V \subset \Omega$. $\therefore$ Ω is open in $X \times Y$ $\square$

* For linear maps between F-spaces the converse is also true.

11.5. Theorem (The closed graph theorem)

Suppose $\Lambda : \mathcal{F}_1 \to \mathcal{F}_2$ is linear and $\mathcal{F}_1, \mathcal{F}_2$ are F-spaces. If $\mathcal{G}(\Lambda)$ is closed in $\mathcal{F}_1 \times \mathcal{F}_2$ (with product topology), then Λ is continuous.

11.6. Proposition If $\mathcal{F}_1$ and $\mathcal{F}_2$ are F-spaces, with compatible invariant metrics d_1 and d_2, then $d((x_1, x_2), (y_1, y_2)) := d_1(x_1, y_1) + d_2(x_2, y_2)$ defines an invariant metric on $\mathcal{F}_1 \times \mathcal{F}_2$ which is compatible with the product topology. In addition, $\mathcal{F}_1 \times \mathcal{F}_2$, with the usual componentwise definition of addition and scalar multiplication, is an F-space.

Proof: Exercise. Note the similarity to direct sum of two Hilbert spaces. $\square$

Proof of 11.5. ① Λ linear $\Rightarrow$ $\mathcal{G}(f)$ is an subspace of $\mathcal{F}_1 \times \mathcal{F}_2$. ($\mathcal{G}(f) = \tilde{\lambda}(\mathcal{F}_1)$, where $\tilde{\lambda}(x) = (x, \Lambda x)$ is obviously linear: $\tilde{\lambda}(\alpha x + \beta y) = (\alpha x + \beta y, \alpha \Lambda x + \beta \Lambda y)$ $= \alpha(x, \Lambda x) + \beta(y, \Lambda y)$. Apply 5.1.b))
As $\mathcal{G}(f)$ is closed $\Rightarrow$ $\mathcal{G}(f)$ is an F-space. ($x = \lim x_k$ $\Rightarrow x \in \{x_k\}$.)

② Let $P_1 : \mathcal{F}_1 \times \mathcal{F}_2 \to \mathcal{F}_1$ and $P_2 : \mathcal{F}_1 \times \mathcal{F}_2 \to \mathcal{F}_2$ denote the component projection maps & $P_i(x_1, x_2) = x_i$. Define $\pi_1 = P_1|_{\mathcal{G}(f)} : \mathcal{G}(f) \to \mathcal{F}_1$, i.e., set $\pi_1(x, \Lambda x) = x$ $\forall x \in \mathcal{F}_1$. Then π_1 is obviously a linear bijection. If $U \in \mathcal{T}_{\mathcal{F}_1}$ $\Rightarrow$ $\pi_1^{\leftarrow}(U) = (U \times \mathcal{F}_2) \cap \mathcal{G}(f)$ is open in $\mathcal{G}(f)$. Thus π_1 is continuous. If V is open in $\mathcal{G}(f)$ and $x \in \pi_1(V)$ $\Rightarrow$ $\exists U_1 \in \mathcal{T}_{\mathcal{F}_1}$, $U_2 \in \mathcal{T}_{\mathcal{F}_2}$ s.t. with $y = \Lambda x$, $(x,y) \in V$, $x \in U_1$, $y \in U_2$, $U_1 \times U_2 \subset V$. $\Rightarrow$ $\pi_1(U_1 \times U_2) = U_1 \subset \pi_1(V)$ and $x \in U_1$. $\therefore$ $\pi_1(V)$ is open. Thus π_1 is also open mapping. We can apply Corollary 11.2.a) to π_1 $\Rightarrow$ $\pi_1^{-1} : \mathcal{F}_1 \to \mathcal{G}(f)$ is continuous. As obviously $\Lambda = P_2|_{\mathcal{G}(f)} \circ \pi_1^{-1}$ and P_2 is (by def.) continuous in the product topology $\Rightarrow$ $P_2|_{\mathcal{G}(f)}$ is contin. $\Rightarrow$ Λ is continuous. $\square$

11.7. Proposition: If $\Lambda : \mathcal{F}_1 \to \mathcal{F}_2$ is linear and $\mathcal{F}_1, \mathcal{F}_2$ are F-spaces, then $\mathcal{G}(\Lambda)$ is closed iff [If $(x_n)_{n \in \mathbb{N}_+} \subset \mathcal{F}_1$ is s.t. $\exists x = \lim_{n \to \infty} x_n$ and $\exists y = \lim_{n \to \infty} \Lambda x_n$, then $y = \Lambda x$.]

Proof: "$\Leftarrow$": Suppose [...] is true. Consider $(x,y) \in \overline{\mathcal{G}(\Lambda)}$.
$\Rightarrow$ $\forall n \in \mathbb{N}_+$ $\exists x_n \in \mathcal{F}_1$ s.t. $d((x,y), (x_n, \Lambda x_n)) < \frac{1}{n}$
$\Rightarrow$ $d_1(x, x_n) + d_2(y, \Lambda x_n) \xrightarrow[n \to \infty]{} 0$ $\Rightarrow$ $x \to x_n$ and $\Lambda x_n \to y$.
Thus by [...] $\Rightarrow$ $y = \Lambda x$ $\Rightarrow$ $(x,y) \in \mathcal{G}(\Lambda)$. $\therefore$ $\mathcal{G}(\Lambda)$ closed.
"$\Rightarrow$": Suppose $\mathcal{G}(\Lambda)$ closed, and (x_n) is s.t. $x_n \to x$ and $\Lambda x_n \to y$. Thus if $\varepsilon > 0$ $\exists N \in \mathbb{N}_+$ s.t.
$d_1(x, x_n) < \frac{\varepsilon}{2}$, $d_2(y, \Lambda x_n) < \frac{\varepsilon}{2}$ $\Rightarrow$
$d((x,y), (x_n, \Lambda x_n)) < \varepsilon$ $\Rightarrow$ $\mathcal{G}(\Lambda) \cap (B((x,y), \varepsilon)) \neq \emptyset$.
Thus $(x,y) \in \overline{\mathcal{G}(\Lambda)} = \mathcal{G}(\Lambda)$ $\Rightarrow$ $y = \Lambda x$. $\square$

11.8. Definitions: a) Let $(X_i)_{i=0}^{N}$, $N \in \mathbb{N}_+$, $N \geq 2$, be vector spaces. A map $T : \prod_{i=1}^{N} X_i \to X_0$ is called <u>multilinear</u>, if all of the maps $T_x^{(i)} : X_i \to X_0$, $i=1,..,N$, $x \in \prod_{j=1}^{N} X_j$

defined by $T_x^{(i)}(y) := T(x_1, x_2, \dots, \overset{i}{y}, \dots) = T(x|_{x_i \to y})$, for $y \in X_i$, are linear.

b) If X_i have topologies, $T : \prod_{i=1}^{N} X_i \to X_0$ is called <u>separately continuous</u> if every $T_x^{(i)}$, $i=1,..,N$, $x \in \prod_{j=1}^{N} X_j$, is continuous.

* If T is continuous w.r.t. the product topology $\Rightarrow$ T is separately continuous.

(Proof. $U \in \mathcal{T}_{X_0} \Rightarrow T^{\leftarrow}U$ open in $\prod X_j$. If $y \in (T_x^{(i)})^{\leftarrow} U$
$\Rightarrow T_x^{(i)}(y) = T(x|_{x_i \to y}) \in U$
$\Rightarrow x|_{x_i \to y} \in T^{\leftarrow} U \Rightarrow \exists\, U_j$, $j=1,..,N$, s.t. $x_j \in U_j \in \mathcal{T}_{X_j}$ $\forall j \neq i$ and $y \in U_i \in \mathcal{T}_{X_i}$ and $\prod U_j \subset T^{\leftarrow} U$.
But then if $y' \in U_i \Rightarrow x|_{x_i \to y'} \in \prod_{j=1}^{N} U_j \subset T^{\leftarrow} U$
$\Rightarrow T_x^{(i)}(y') = T(x|_{x_i \to y'}) \in U$. Therefore, $y \in U_i \subset (T_x^{(i)})^{\leftarrow} U$.
$\therefore (T_x^{(i)})^{\leftarrow} U$ is open. $\therefore T_x^{(i)}$ is continuous.)

* In some cases, the converse holds:

11.9. Theorem: Suppose $T : \mathcal{F}_1 \times V_2 \to V_0$ is multilin. and separately continuous, where $\mathcal{F}_1$ = F-space and V_2, V_0 are topol. vect. spaces. Then $T(x_n, y_n) \xrightarrow[n\to\infty]{} T(\bar{x}, \bar{y})$ whenever $x_n \xrightarrow[n\to\infty]{} \bar{x}$ and $y_n \xrightarrow[n\to\infty]{} \bar{y}$.
(i.e. T is sequentially continuous.)

<u>Proof.</u> Suppose $x_n \in \mathcal{F}_1$, $y_n \in V_2$, $n \in \mathbb{N}_+$, are s.t. $x_n \to \bar{x}$ and $y_n \to \bar{y}$. Define $b_n : \mathcal{F}_1 \to V_0$, $n \in \mathbb{N}_+$, by setting $b_n(x) := T(x, y_n)$. By assumption, each b_n is linear and contin. For a fixed x, so is the map $y \mapsto T(x,y)$ and thus $b_n(x) \xrightarrow{n\to\infty} T(x, \bar{y})$.
Thus the orbit $\{b_n(x)\}_{n \in \mathbb{N}_+}$ is Cauchy $\overset{8.1.e)}{\Rightarrow}$ bounded in V_0.
By 10.8.a) $\Rightarrow \{b_n\}_n$ is equicontinuous.
If $0 \in W \in \mathcal{T}_{V_0} \Rightarrow \exists\, U \in \mathcal{T}_{V_0}$ s.t. $0 \in U$, $U + U \subset W$.
By equicont. $\Rightarrow \exists\, V \in \mathcal{T}_{\mathcal{F}_1}$ s.t. $0 \in V$ and $b_n(V) \subset U$ $\forall n \in \mathbb{N}_+$.
Now $T(\bar{x}, \bar{y}) - T(x_n, y_n) = T(\bar{x}, \bar{y} - y_n) + b_n(\bar{x} - x_n)$.

By assumption, $\exists N \in \mathbb{N}_+$ s.t. $x_n \in \bar{x} - V$ and, as $y \mapsto T(\bar{x}, y)$ is contin. & lin. $\Rightarrow T(\bar{x}, \bar{y} - y_n) \to T(\bar{x}, 0) = 0$, also $T(\bar{x}, \bar{y} - y_n) \in U$, for all $n \geq N$. Then $\forall n \geq N$ we have $T(\bar{x}, \bar{y}) - T(x_n, y_n) \in U + b_n(r) \subset U + U \subset W$. This proves that $T(x_n, y_n) \to T(\bar{x}, \bar{y})$ $\square$

11.10. Proposition: a) If $T: \mathcal{F}_1 \times \mathcal{F}_2 \to \mathcal{V}_0$ is multilin. and separately continuous, where $\mathcal{F}_1, \mathcal{F}_2$ are F-spaces and $\mathcal{V}_0$ is a topol. vect. space, then T is continuous.

b) If $T: \prod_{j=1}^N \mathcal{B}_j \to \mathcal{B}_0$, with $\mathcal{B}_i =$ Banach $\forall i = 0, 1, \dots, N$, is multilin. and separately continuous, then $\exists M \geq 0$ s.t. $\| T(x_1, \dots, x_N) \| \leq M \prod_{j=1}^N \| x_j \|$ $\forall x_i \in \mathcal{B}_i$, for $i = 1, 2, \dots, N$

c) A conjugate multilinear map $T: \prod_{i=1}^N \mathcal{H}_i \to \mathbb{C}$, $N \in \mathbb{N}_+$, belongs to $\bigotimes_{i=1}^N \mathcal{H}_i$ if and only if

i) $\sup_{\| x_i \| = 1 \, \forall i} | T x | < \infty$

and ii) For some collection of ONBs $(e_i(\ell))_{\ell \in I_i}$ for $\mathcal{H}_i$, $i = 1, 2, \dots, N$, we have

$$\sum_{\ell \in I = \prod_{i=1}^N I_i} | T(e_1(\ell_1), \dots, e_N(\ell_N)) |^2 < \infty.$$

Proof. a) Since $\mathcal{F}_1 \times \mathcal{F}_2$ is metrizable, sequential continuity $\Rightarrow$ continuity. (If $Tx \in U \in \mathcal{T}_{\mathcal{V}_0}$ and $\forall n \in \mathbb{N}_+$ $B(x, \frac{1}{n}) \not\subset T^{\leftarrow} U \Rightarrow \exists x_n \in \mathcal{F}_1 \times \mathcal{F}_2$ s.t. $d(x, x_n) < \frac{1}{n}$ and $Tx_n \notin U \Rightarrow x_n \to x$ but $Tx_n \not\to Tx$ ↯ $\therefore \forall x \in T^{\leftarrow} U \; \exists n \in \mathbb{N}_+$ s.t. $B(x, \frac{1}{n}) \subset T^{\leftarrow} U$. $\therefore T^{\leftarrow} U$ is open.)

b) Exercise.

c) Suppose $T: \prod_{i=1}^N \mathcal{H}_i \to \mathbb{C}$ is conj. multilin. and i) & ii) hold. Denote $\lambda(\ell) = T(e_1(\ell_1), \dots, e_N(\ell_N)) \in \mathbb{C}$ for $\ell \in I := \prod_{i=1}^N I_i$. (Since $\sum_{\ell \in I} |\lambda(\ell)|^2 < \infty$

$\Rightarrow \forall i \; \exists I_{0,i} \subset I_i$ which is countable and for $\ell \notin I_0 := \prod_{i=1}^N I_{0,i}$ we have $\lambda(\ell) = 0$.) If $u_i \neq 0 \; \forall i$

By i) & conj. multilin. $\Rightarrow | T(u_1, \dots, u_N) | \leq \prod_{i=1}^N \| u_i \| \, M$ (*)

where $M := \sup_{\|x_i\|=1\,\forall i} |Tx| < \infty$. If $\psi_i = 0$ for some i,

$\Rightarrow |T(\psi_1, \dots, \psi_N)| = 0 \leq M \cdot 0 = M \prod_{i=1}^{N} \|\psi_i\|$.
Thus $(*)$ holds $\forall\, \Psi \in \prod_{i=1}^{N} \mathcal{H}_i \Rightarrow T$ is separately continuous. If $\psi \in \pi \mathcal{H}_i \underset{\text{ONB}}{\Rightarrow} \forall i$ we can write

$$\psi_i = \sum_{l \in \tilde{I}_{i,0}} (e_i(l), \psi_i)\, e_i(l) \quad \text{where } \tilde{I}_{i,0} \subset I_i \text{ is countable.} \qquad (2.17)$$

Thus by separate continuity and conj. multilinearity

$$T(\psi_1, \dots, \psi_N) = \sum_{l_1 \in \tilde{I}_{1,0}} \Big(\sum_{l_2 \in \tilde{I}_{2,0}} \Big(\cdots \sum_{l_N \in \tilde{I}_{N,0}} \prod_{i=1}^{N} (\psi_i, e_i(l)) \cdot \lambda(l) \Big) \cdots \Big)$$

Letting $I_0 := \prod_{i=1}^{N} \tilde{I}_{i,0}$ we have here $\sum_{l \in \tilde{I}_0} \Big| \prod_{i=1}^{N} (\psi_i, e_i(l)) \Big|^2$
$= \prod_{i=1}^{N} \|\psi_i\|^2 < \infty$ and as also $\sum_{l \in \tilde{I}_0} |\lambda(l)|^2 < \infty$

we can use Fubini and conclude that

$$T(\psi_1, \dots, \psi_N) = \sum_{l \in \tilde{I}_0} \lambda(l) \Big(\bigotimes_{i=1}^{N} \psi_i, \bigotimes_{i=1}^{N} e_i(l) \Big)$$

here summation can be made over all $l \in I$ since, if $l \notin \tilde{I}_0 \Rightarrow \exists i_0$ s.t. $l_{i_0} \notin \tilde{I}_{i_0,0} \Rightarrow (e_{i_0}(l_{i_0}), \psi_{i_0}) = 0$
$\Rightarrow (\otimes \psi_i, \otimes e_i(l)) = 0$. But since $\sum_{l \in I} |\lambda(l)|^2 < \infty$ and $\otimes e_i(l)$ is an ONB for $\otimes \mathcal{H}_i$
$\overset{2.17}{\Rightarrow} u = \sum_{l \in I} \lambda(l) \bigotimes_{i=1}^{N} e_i(l) \in \bigotimes_{i=1}^{N} \mathcal{H}_i$ and

$$\Big(\bigotimes_{i=1}^{N} \psi_i, u \Big) = \sum_{l \in I} \lambda(l) \Big(\bigotimes_{i=1}^{N} \psi_i, \bigotimes_{i=1}^{N} e_i(l) \Big) = T(\psi_1, \dots, \psi_N)$$

This was the identification we made before $\Rightarrow T \hat{=} u \in \otimes \mathcal{H}_i$
"$\Rightarrow$" Suppose $u \in \bigotimes_{i=1}^{N} \mathcal{H}_i$ and consider $T: \prod_{i=1}^{N} \mathcal{H}_i \to \mathbb{C}$

defined by $T(\psi_1, \dots, \psi_N) := \Big(\bigotimes_{i=1}^{N} \psi_i, u \Big)$. T is clearly conj. mult. lin. and for any ONBs $\sum_{l \in I} |T(e(l))|^2$

$$= \sum_{l \in I} \Big| \Big(\bigotimes_{i=1}^{N} e_i(l_i), u \Big) \Big|^2 \overset{2.17}{=} \|u\|^2 < \infty, \Rightarrow ii) \text{ holds.}$$

If $\phi_i \in \mathcal{H}_i$ and $\hat{\psi}_j = \begin{cases} \psi_j, & j \neq i \\ \phi_i, & j = i \end{cases}$ then

$$\bigotimes_{j=1}^{N} \hat{\psi}_j - \bigotimes_{j=1}^{N} \psi_j = \bigotimes_{j=1}^{N} \psi_j' \quad \text{where } \psi_j' = \begin{cases} \psi_j, & j \neq i \\ \phi_i - \psi_i, & j = i \end{cases}$$

$$\Rightarrow \quad \| \otimes \hat{\psi} - \otimes \psi \|^2 = \prod_{j=1}^{N} \| \psi_j' \|^2$$

$$= \prod_{j \neq i} \| \psi_j \|^2 \cdot \| \phi_i - \psi_i \|^2 . \quad \text{Let } \overline{T}(\psi) = T(\psi)^* .$$

$$\Rightarrow \quad | \overline{T}(\psi|_{\psi_i \to \phi_i}) - \overline{T}(\psi) | = | (\bigotimes_{j=1}^{N} \hat{\psi}_j - \bigotimes_{j=1}^{N} \psi_j , u)^* |$$

$$\leq \|u\| \, \| \otimes \psi' \| = \|u\| \prod_{j \neq i} \| \psi_j \| \cdot \| \phi_i - \psi_i \|$$

$\Rightarrow$ $\overline{T}$ is separately contin. and multilin. By b) $\Rightarrow \exists M \geq 0$ s.t. $\forall \psi_i \in \mathcal{H}_i$; $| \overline{T}(\psi) | \leq M \prod_{j=1}^{N} \| \psi_j \|$.

Thus if $\| \psi_i \| = 1 \; \forall i \Rightarrow | T(\psi) | = | \overline{T}(\psi) | \leq M < \infty$.
$\Rightarrow$ i) holds. □

Appendix to proof of Theorem 11.1. :

* We used here the fact that, if $\Phi : V_1 \to V_2$ is a linear homeomorphism, then (x_n) is Cauchy in V_1 if and only if (Φx_n) is Cauchy in V_2. Thus then V_1 is complete iff V_2 is complete. Without linearity the statement need not be true, as was seen in Ex. 5.2. b).

Proof: (x_n) Cauchy in V_1 and $0 \in V_2 \in \mathcal{T}_{V_2}$
$\Rightarrow 0 \in \Phi^{-1}(V_2) \in \mathcal{T}_{V_1} \Rightarrow \exists N \in \mathbb{N}_+$ s.t. $x_n - x_m \in \Phi^{-1}(V_2) \; \forall n,m \geq N$
Thus if $n, m \geq N \Rightarrow \Phi(x_n) - \Phi(x_m) = \Phi(x_n - x_m) \in V_2$.
$\therefore (\Phi(x_n))$ is Cauchy in V_2.
The other direction follows by applying this to Φ^{-1} which also is a linear homeomorphism. □