10.4. Corollary: If $\mathbb{X}$ is either ① or ②, then $\mathbb{X}$ is of 2nd category in $\mathbb{X}$.

Proof. If $F_n \subset \mathbb{X}$, $n \in \mathbb{N}_+$, are nowhere dense $\Rightarrow V_n := \overline{F}_n^c$ is open and $\overline{V_n} = \mathbb{X}$ (If $U \in \mathcal{T}_{\mathbb{X}}$, $U \neq \emptyset$ $\Rightarrow U \not\subset \overline{F}_n \Rightarrow U \cap V_n = U \cap \overline{F}_n^c \neq \emptyset$.) Thus By 10.3. $\Rightarrow \bigcap_{n \in \mathbb{N}_+} V_n$ is dense in $\mathbb{X} \Rightarrow \bigcap_n V_n \neq \emptyset$

$$\Rightarrow \bigcup_{n \in \mathbb{N}_+} \overline{F}_n = \Big(\bigcap_{n \in \mathbb{N}_+} \overline{F}_n^c\Big)^c \neq \mathbb{X} \quad \square$$

10.5. Definition Let $\mathcal{V}_1, \mathcal{V}_2$ be topol. vect. spaces, and $\Gamma = \{\Lambda_i\}_{i \in I}$ where each Λ_i, $i \in I$, is a linear map $\mathcal{V}_1 \to \mathcal{V}_2$. We say Γ is equicontinuous if $[\, 0 \in W \in \mathcal{T}_{\mathcal{V}_2} \Rightarrow \exists V \in \mathcal{T}_{\mathcal{V}_1}$

s.t. $0 \in V$ and $\Lambda(V) \subset W \;\; \forall \Lambda \in \Gamma \,]$

* If Γ is equicontinuous, as above, then each $\Lambda \in \Gamma$ is continuous. Also, if $\Gamma = \{\Lambda_0\}$, then Γ is equicont. iff Λ_0 is continuous. (Thrm. 5.2)
* Continuous linear maps are bounded (Thrm. 8.3.) The following theorem shows that equicontinuous collection of linear maps is uniformly bounded.

10.6. Theorem Consider an equicontinuous collection Γ of linear maps $\mathcal{V}_1 \to \mathcal{V}_2$, where $\mathcal{V}_1$ and $\mathcal{V}_2$ are topol. vect. spaces. If $E_1 \subset \mathcal{V}_1$ is bounded then $\exists E_2 \subset \mathcal{V}_2$ s.t. E_2 is bounded and $\Lambda(E_1) \subset E_2 \;\; \forall \Lambda \in \Gamma$.

Proof. Suppose $E_1 \subset \mathcal{V}_1$ is bounded, and set $E_2 := \bigcup_{\Lambda \in \Gamma} \Lambda(E_1)$. Consider $0 \in W_2 \in \mathcal{T}_{\mathcal{V}_2}$. $\overset{\Gamma \text{ equicont.}}{\Rightarrow}$

$\exists W_1 \in \mathcal{T}_{\mathcal{V}_1}$ s.t. $0 \in W_1$ and $\Lambda(W_1) \subset W_2 \;\; \forall \Lambda \in \Gamma$. As E_1 bounded $\Rightarrow \exists t_0 > 0$ s.t. $E_1 \subset t W_1 \;\; \forall t > t_0$. Thus if $t > t_0$ and $\Lambda \in \Gamma \Rightarrow \Lambda(E_1) \subset \Lambda(t W_1) = t \Lambda(W_1) \subset t W_2$. Thus $E_2 \subset t W_2 \;\; \forall t > t_0 > 0$. $\therefore E_2$ is bounded $\square$

10.7. Theorem (Banach-Steinhaus)

Consider a collection Γ of linear maps $\mathcal{V}_1 \to \mathcal{V}_2$, where $\mathcal{V}_1$ and $\mathcal{V}_2$ are topol. vect. spaces. Assume that each $\Lambda \in \Gamma$ is continuous. For any $x \in \mathcal{V}_1$ let $\Gamma(x) := \{\Lambda x \mid \Lambda \in \Gamma\} \subset \mathcal{V}_2$ denote the orbit of x under Γ, and define $B := \{x \in \mathcal{V}_1 \mid \Gamma(x) \text{ is bounded}\}$.

If B is of 2nd category in $\mathcal{V}_1$, then actually $B = \mathcal{V}_1$ and Γ is equicontinuous.

Proof. Suppose $0 \in W_2 \in \mathcal{T}_{\mathcal{V}_2}$. Then there are balanced $U, V \in \mathcal{T}_{\mathcal{V}_2}$ s.t. $0 \in U$, $0 \in V$, and $\bar{U} + \bar{U} \subset V \subset W_2$. (Lemma 4.10. and Thrm. 4.9 & 4.12.). Set $E_1 := \bigcap_{\Lambda \in \Gamma} \Lambda^{\leftarrow}(\bar{U})$.

If $x \in B \Rightarrow \Gamma(x)$ is bounded $\Rightarrow \exists\, n \in \mathbb{N}_+$ s.t. $\Gamma(x) \subset nU$ $\Rightarrow x \in \Lambda^{\leftarrow}(nU) = n\Lambda^{\leftarrow}U \;\; \forall \Lambda \in \Gamma \Rightarrow x \in nE_1$.

Thus $B \subset \bigcup_{n \in \mathbb{N}_+} (nE_1)$. If all nE_1, $n \in \mathbb{N}_+$, were of 1st category in $\mathcal{V}_1$, then their union would be of 1st cat., and thus also B. (Prop. 10.2.) $\lightning$

Thus $\exists\, n_0 \in \mathbb{N}_+$ s.t. $n_0 E_1$ is of 2nd cat. in $\mathcal{V}_1$. Since $x \mapsto n_0 x$ is a homeo, this implies that E_1 is of 2nd cat. As each $\Lambda \in \Gamma$ is contin. $\Rightarrow \Lambda^{\leftarrow}\bar{U}$ is closed. Thus E_1 closed $\Rightarrow E_1 = \bar{E}_1$ has an interior point x_0. $\Rightarrow \exists\, W_1 \in \mathcal{T}_{\mathcal{V}_1}$ s.t. $0 \in W_1$ and $x_0 + W_1 \subset E_1$. Then for any $\Lambda \in \Gamma$, $y \in \Lambda W_1 \Rightarrow \exists\, x \in E_1$ s.t. $y = \Lambda(x - x_0) = \Lambda x - \Lambda x_0 \in \bar{U} - \bar{U}$ $= \bar{U} + \bar{U} \subset V \subset W_2$ $\therefore \Lambda W_1 \subset W_2 \;\; \forall \Lambda \in \Gamma$ and thus Γ is equicontin. For any $x \in \mathcal{V}_1$ the singlet $\{x\}$ is compact $\Rightarrow$ bounded in $\mathcal{V}_1$, and thus by Thrm. 10.6., $\exists\, E_2 \subset \mathcal{V}_2$ bounded s.t. $\Lambda x \subset E_2 \;\; \forall \Lambda \in \Gamma$ $\Rightarrow \Gamma(x) \subset E_2 \Rightarrow \Gamma(x)$ is bounded. This shows that $B = \mathcal{V}_1$. $\square$

10.8. Corollaries (Cases when pointwise boundedness implies uniform boundedness.)

(a) If Γ is a collection of continuous linear maps $\mathcal{F}_1 \to \mathcal{V}_2$, where $\mathcal{F}_1$ is an F-space and $\mathcal{V}_2$ is a topol. vect. space, and all orbits $\Gamma(x)$, $x \in \mathcal{F}_1$, are bounded in $\mathcal{V}_2$, then Γ is equicontinuous.

(b) Suppose Γ is a collection of linear maps $\mathcal{B}_1 \to \mathcal{B}_2$, between Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$.

If $\sup_{\Lambda \in \Gamma} \|\Lambda x\| < \infty \ \forall x \in \mathcal{B}_1$ and $\sup_{\|x\|=1} \|\Lambda x\| < \infty \ \forall \Lambda \in \Gamma$,

then $\exists M \geq 0$ s.t. $\|\Lambda x\| \leq M \|x\| \ \forall x \in \mathcal{B}_1, \Lambda \in \Gamma$.

Proof. (a) Now $B = \mathcal{F}_1$ = complete metric space $\overset{\text{Baire}}{\Rightarrow}$ B is of 2nd cat. $\Rightarrow$ Γ is equicont. ∎

(b) If $x \in \mathcal{B}_1 \Rightarrow \Gamma(x) = \{\Lambda x \mid \Lambda \in \Gamma\} \underset{\text{assumpt.}}{\Rightarrow} \exists m(x) \geq 0$ s.t. $\|y\| \leq m(x) < \infty \ \forall y \in \Gamma(x)$ $\Rightarrow \Gamma(x)$ bounded. Also $\|\Lambda\| < \infty \ \forall \Lambda \in \Gamma$ $\Rightarrow$ every $\Lambda \in \Gamma$ is continuous. Since $\mathcal{B}_1$ = Banach = F-space, by (a) $\Rightarrow \Gamma$ is equicontinuous. Since $B_1 = \{x \mid \|x\| \leq 1\}$ is bounded, we thus can find $M \geq 0$ s.t. $\|\Lambda B_1\| \leq M \ \forall \Lambda \in \Gamma$. (Thrm. 10.6.) Thus if $x \neq 0$, $\Lambda \in \Gamma$, then $\|\Lambda x\| = \|x\| \, \|\Lambda \frac{x}{\|x\|}\| \leq M \|x\|$, and $\|\Lambda 0\| = 0 \leq M \cdot \|0\|$ □

10.9. Proposition

Consider $\Lambda_n : \mathcal{V}_1 \to \mathcal{V}_2$, $n \in \mathbb{N}_+$, where $\mathcal{V}_1, \mathcal{V}_2$ are topol. vect. spaces and each Λ_n is continuous and linear. Define $E := \{x \in \mathcal{V}_1 \mid (\Lambda_n x)_{n \in \mathbb{N}_+}$ is a (topological) Cauchy sequence$\}$. E is a subspace, and, if E is of 2nd category in $\mathcal{V}_1$, then $E = \mathcal{V}_1$, and the collection $(\Lambda_n)_{n \in \mathbb{N}_+}$ is equicontinuous.

* Reminder: $(x_n)_{n \in \mathbb{N}_+}$ is a (topological) Cauchy sequence in $\mathcal{V}$ = T.V.S. iff
$[\, 0 \in V \in \mathcal{T}_{\mathcal{V}} \Rightarrow \exists N \in \mathbb{N}_+$ s.t. $x_n - x_m \in V \ \forall n, m \geq N \,]$

* $(x_n)_{n \in \mathbb{N}_+}$ convergent $\Rightarrow$ (x_n) Cauchy; If $x = \lim x_n$, $0 \in V \in \mathcal{T}_{\mathcal{V}} \Rightarrow \exists W \in \mathcal{T}_{\mathcal{V}}$ balanced, $0 \in W$, $W + W \subset V$ $\Rightarrow$

$\exists N \in \mathbb{N}_+$ s.t. $\forall n \geq N$: $x_n \in x + W$. Thus then
$x_n - x_m = (x_n - x) - (x_m - x) \in W - W = W + W \subset V$ $\forall n, m \geq N$. □

* If $\mathcal{T}_N$ induced by invar. metric d, then d-Cauchy $\Leftrightarrow$ topol. Cauchy.

<u>Proof.</u> ② $x \in E \Rightarrow (\Lambda_n x)$ Cauchy $\overset{8.1\,c)}{\Rightarrow} \{\Lambda_n x\}_n$ bounded.
Since $\{\Lambda_n x\}_{n \in \mathbb{N}_+}$ is the orbit of x, we have
$E \subset B$ in Thrm. 10.7. $\Rightarrow$ B is 2nd cat.
$\underset{10.7.}{\Rightarrow}$ $\{\Lambda_n\}_{n \in \mathbb{N}_+}$ is equicontinuous.

① Since $\Lambda_n 0 = 0$ $\forall n$ $\Rightarrow$ $0 \in E$. If $\alpha, \beta \in \mathbb{k}$, $\alpha \neq 0$ or $\beta \neq 0$,
$x, y \in E$ and $0 \in V \in \mathcal{T}_{N_2} \Rightarrow \exists N \in \mathbb{N}_+$ s.t.
$\Lambda_n x - \Lambda_m x \in W$ and $\Lambda_n y - \Lambda_m y \in W$ $\forall n, m \geq N$
where $W \in \mathcal{T}_{N_2}$ is such that $0 \in W$, W is balanced,
and $W + W \subset \frac{1}{\max(|\alpha|, |\beta|)} V$ $(\in \mathcal{T}_{N_2})$ $\Rightarrow$ $\forall n, m \geq N$:

$$\Lambda_n(\alpha x + \beta y) - \Lambda_m(\alpha x + \beta y) = \alpha(\Lambda_n x - \Lambda_m x) + \beta(\Lambda_n y - \Lambda_m y)$$
$$\in \alpha W + \beta W = |\alpha| W + |\beta| W \subset \max(|\alpha|, |\beta|)[W + W] \subset V$$

$\therefore$ $\alpha x + \beta y \in E$. If $\alpha = 0 = \beta \Rightarrow \alpha x + \beta y = 0 \in E$.
$\therefore$ E is subspace □

③ Now E subspace $\overset{4.11.c)}{\Rightarrow}$ $\bar{E}$ is subspace. But $\bar{E}$
is also 2nd cat. $\Rightarrow \exists x_0 \in E$ and $V_0 \in \mathcal{T}_N$ s.t. $0 \in V_0$,
$x_0 + V_0 \subset \bar{E}$. By 4.14. a) $N_1 = \bigcup_{n=1}^{\infty} (n V_0)$, thus
$x \in N_1 \Rightarrow \exists n \in \mathbb{N}_+, y \in V_0$ s.t. $x = ny = n(\underbrace{y + x_0}_{\in \bar{E}}) - n\underbrace{x_0}_{\in \bar{E}}$
$\in \bar{E}$ as $\bar{E}$ is a subspace.
Thus $N_1 = \bar{E}$, i.e. E is a dense subspace.
Fix $x \in N_1$. Consider $0 \in W_2 \in \mathcal{T}_{N_2}$. By equicontinuity,
$\exists W_1 \in \mathcal{T}_{N_1}$ s.t. $0 \in W_1$ and $\Lambda_n(W_1) \subset W_2$ $\forall n$. As E
is dense $\exists x' \in (x + W_1) \cap E \Rightarrow \exists N \in \mathbb{N}_+$ s.t.
$\Lambda_n x' - \Lambda_m x' \in W_2$ $\forall n, m \geq N$. Then $\forall n, m \geq N$,
$\Lambda_n x - \Lambda_m x = \Lambda_n(x - x') + \Lambda_n x' - \Lambda_m x' + \Lambda_m(x' - x)$
$\in -\Lambda_n(W_1) + W_2 + \Lambda_m W_1 \subset -W_2 + W_2 + W_2$.
If $0 \in V_2 \in \mathcal{T}_{N_2}$, $\exists W_2 \in \mathcal{T}_{N_2}$ s.t. $0 \in W_2$, W_2 balanced,
and $W_2 + W_2 + W_2 + W_2 \subset V_2$. Thus the above
computation shows that then $\exists N \in \mathbb{N}_+$ s.t. $\forall n, m \geq N$
$\Lambda_n x - \Lambda_m x \in V_2$. Thus $x \in E$, and therefor
$E = N_1$ □

10.10 Corollaries (Convergent sequences of cont. lin. maps)

(a) Consider $\Lambda_n : \mathcal{V}_1 \to \mathcal{F}_2$, $n \in \mathbb{N}_+$, where $\mathcal{V}_1$ is a topol. vect. space, $\mathcal{F}_2$ is an F-space, and each Λ_n is continuous and linear. Define $\Lambda : E \to \mathcal{F}_2$ by $\Lambda x = \lim_{n\to\infty} \Lambda_n x$, where $E = \{x \in \mathcal{V}_1 \mid \exists \lim_{n\to\infty} \Lambda_n x\}$.

If E is of 2nd category in $\mathcal{V}_1$, then $E = \mathcal{V}_1$ and Λ is continuous and linear.

(b) Consider $\Lambda_n : \mathcal{F}_1 \to \mathcal{V}_2$, $n \in \mathbb{N}_+$, where $\mathcal{F}_1$ is an F-space, $\mathcal{V}_2$ is a topol. vect. space, and each Λ_n is continuous and linear. If $\forall x \in \mathcal{F}_1$ $\exists \lim_{n\to\infty} \Lambda_n x$, then $\Lambda x := \lim_{n\to\infty} \Lambda_n x$ defines a continuous and linear map $\Lambda : \mathcal{F}_1 \to \mathcal{V}_2$.

__Proof__: (a) As $\mathcal{F}_2$ is complete $\Rightarrow E = \{x \in \mathcal{V}_1 \mid (\Lambda_n x) \text{ Cauchy}\}$. By 10.9., then $E = \mathcal{V}_1$, and $\Lambda : \mathcal{V}_1 \to \mathcal{F}_2$. Since $\mathcal{F}_2$ is T.V.S., $\alpha \lim_n y_n + \beta \lim_n y'_n = \lim_n (\alpha y_n + \beta y'_n)$. Thus $\Lambda(\alpha x + \beta x') = \lim_n \Lambda_n(\alpha x + \beta x') = \lim (\alpha \Lambda_n x + \beta \Lambda_n x')$ $= \alpha \Lambda(x) + \beta \Lambda(x')$. Thus Λ is linear, and by 5.2. it suffices to prove that it is contin. at 0. If $0 \in W_2 \in \mathcal{T}_{\mathcal{F}_2}$, $\exists W \in \mathcal{T}_{\mathcal{F}_2}$ s.t. $0 \in W$ and $\overline{W} \subset W_2$. By equicontin. of $\{\Lambda_n\}$, $\exists W_1 \in \mathcal{T}_{\mathcal{V}_1}$ s.t. $0 \in W_1$ and $\Lambda_n(W_1) \subset W$ $\forall n$. $\Rightarrow$ if $x \in W_1$, then $\Lambda x = \lim_{n\to\infty} \underbrace{\Lambda_n x}_{\in W} \in \overline{W}$.

Thus $\Lambda(W_1) \subset \overline{W} \subset W_2$. $\therefore$ Λ is continuous ∎

(b) If $x \in \mathcal{F}_1$, the seq. $(\Lambda_n x)$ converges $\overset{8.1.e)}{\Rightarrow}$ it is bounded. Thus by 10.8.a) $\Rightarrow (\Lambda_n)_{n\in\mathbb{N}_+}$ is equicontin. Exactly as above, this implies that $\Lambda : \mathcal{F}_1 \to \mathcal{V}_2$ is continuous at 0. As $\mathcal{V}_2$ is a topol. vect. space, we can conclude that Λ is linear, as above. $\Rightarrow$ Λ is also continuous. □