Appendix: set theory

If X, Y, and I are sets, and $A_i \subset X$, $B_i \subset Y$, for $i \in I$, then all of the following hold:

De Morgan's laws: (Here $A^c = X \setminus A$, $B^c = Y \setminus B$ denote complements.)

$$\begin{aligned} \Big(\bigcup_{i\in I} A_i\Big)^c &= \bigcap_{i\in I} A_i^c\,, \\ \Big(\bigcap_{i\in I} A_i\Big)^c &= \bigcup_{i\in I} A_i^c\,, \end{aligned}$$

and also

$$A \cap \Big(\bigcup_{i\in I} A_i\Big) = \bigcup_{i\in I} (A \cap A_i)\,.$$

If, in addition, $f : X \to Y$, then

$$\begin{aligned} A &\subset f^{\leftarrow}(f(A)) \\ f(f^{\leftarrow}(B)) &= B \cap f(X) \\ f^{\leftarrow}(B^c) &= (f^{\leftarrow}(B))^c \\ f(A \cap f^{\leftarrow}(B)) &= f(A) \cap B \\ f\Big(\bigcup_{i\in I} A_i\Big) &= \bigcup_{i\in I} f(A_i) \\ f^{\leftarrow}\Big(\bigcup_{i\in I} B_i\Big) &= \bigcup_{i\in I} f^{\leftarrow}(B_i) \\ f^{\leftarrow}\Big(\bigcap_{i\in I} B_i\Big) &= \bigcap_{i\in I} f^{\leftarrow}(B_i) \\ f\Big(\bigcap_{i\in I} A_i\Big) &\subset \bigcap_{i\in I} f(A_i) \end{aligned}$$

If f is *one-to-one* (injective), then also

$$f\Big(\bigcap_{i\in I} A_i\Big) = \bigcap_{i\in I} f(A_i) \quad \text{and} \quad A = f^{\leftarrow}(f(A))\,.$$

9. Quotient spaces

9.1. Definition: Let N be a subspace of a vector space X. Then $x \sim y :\Leftrightarrow x-y \in N$ defines an equivalence relation on X, and the following notations will be used henceforth:

(a) $X/N := X/\sim$ = quotient space of X modulo N

(b) $\pi_N : X \to X/N$ defined by $\pi_N(x) = [x]$ is called the corresponding quotient map.

(c) For $[x], [y] \in X/N$, $\alpha \in \mathbb{K}$, we define $[x]+[y] := [x+y]$ and $\alpha[x] := [\alpha x]$.

9.2. Proposition: a) $[x] = x + N$

b) X/N is a vector space, with zero $[0] = N$.

c) π_N is linear, and $\ker \pi_N = N$.

Proof: Straightforward linear algebra, skipped. □

9.3. Theorem: Let V be a topol. vect. space, and assume $N \subset V$ is a closed subspace. Define $\mathcal{T}_{V/N} := \{ W \subset V/N \mid \pi^{\leftarrow} W \in \mathcal{T}_V \}$. Then

(a) $\mathcal{T}_{V/N}$ is a topology on V/N and it makes V/N into a topol. vect. space.

The following staments are then true w.r.t. $\mathcal{T}_V$ and $\mathcal{T}_{V/N}$.

(b) π_N is continuous and open

(c) If $\mathcal{B}$ is a local base for $\mathcal{T}_V$, then $\{\pi(V)\}_{V \in \mathcal{B}}$ is a local base for $\mathcal{T}_{V/N}$.

(d) If d is an invariant metric on V, compatible with $\mathcal{T}_V$, then

$$\tilde{d}(\pi(x), \pi(y)) := \inf \{ d(x-y, z) \mid z \in N \}$$

is an invariant metric on V/N and it induces $\mathcal{T}_{V/N}$.

(e) If $\|\cdot\|_{\mathcal{V}}$ is a norm on $\mathcal{V}$, compatible with $\mathcal{T}_{\mathcal{V}}$, then

$$\|\pi(x)\|_{\mathcal{V}/N} := \inf\{\|x-z\| \mid z\in N\}$$

defines a norm on $\mathcal{V}/N$, compatible with $\mathcal{T}_{\mathcal{V}/N}$.

(f) If $\mathcal{V}$ satisfies any of the following properties, then so does $\mathcal{V}/N$:
local convexity, local boundedness, metrizable, normable, F-space, Fréchet space or Banach space.

* Terminology: $\mathcal{T}_{\mathcal{V}/N}$ is called the quotient topology, $\tilde{d}$ the quotient metric, and $\|\cdot\|_{\mathcal{V}/N}$ the quotient norm.

Proof: (a & b) One can check using the definitions that for any family of subsets $(A_i)_{i\in I}$, $A_i\subset \mathcal{V}/N$, we have $\pi^{\leftarrow}(\bigcap_{i\in I} A_i) = \bigcap_{i\in I}\pi^{\leftarrow}A_i$ and $\pi^{\leftarrow}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I}\pi^{\leftarrow}A_i$

Since $\pi^{\leftarrow}\phi = \phi$ and $\pi^{\leftarrow}(\mathcal{X}/N) = \mathcal{X}$, it easily follows that $\mathcal{T}_{\mathcal{V}/N}$ is a topology on $\mathcal{V}/N$.
As $\pi^{\leftarrow}(\{[x]\}) = \{y\in\mathcal{V} \mid [y]=[x]\} = x+N$ is closed in $\mathcal{T}_{\mathcal{V}}$, it follows that all singlets are closed in $\mathcal{T}_{\mathcal{V}/N}$. $(\pi^{\leftarrow}(A^c) = (\pi^{\leftarrow}A)^c)$
By def. of $\mathcal{T}_{\mathcal{V}/N}$, π_N is continuous. If $V\in\mathcal{T}_{\mathcal{V}}$, $x\in\pi^{\leftarrow}(\pi(V)) \Leftrightarrow \pi(x)\in\pi(V) \Leftrightarrow \exists y\in V$ s.t. $x\sim y$ $\Leftrightarrow x\in\bigcup_{y\in V}(y+N)$. Thus $\pi^{\leftarrow}(\pi(V)) = V+N\in\mathcal{T}_{\mathcal{V}}$.
$\Rightarrow \pi(V)\in\mathcal{T}_{\mathcal{V}/N}$. This proves that π is open.
To prove continuity of addition, suppose $U\in\mathcal{T}_{\mathcal{V}/N}$ and $x,y\in\mathcal{V}$ are s.t. $[x]+[y]\in U$. Then $\pi^{\leftarrow}U\in\mathcal{T}_{\mathcal{V}}$ and $x+y\in\pi^{\leftarrow}U \Rightarrow \exists V_x, V_y\in\mathcal{T}_{\mathcal{V}}$ s.t. $V_x+V_y\subset\pi^{\leftarrow}U$
& $x\in V_x, y\in V_y \Rightarrow \pi(V_x), \pi(V_y)\in\mathcal{T}_{\mathcal{V}/N}$, $[x]\in\pi(V_x)$ $[y]\in\pi(V_y)$ and $\pi(V_x)+\pi(V_y) \underset{\pi \text{ lin.}}{=} \pi(V_x+V_y)\subset U$. ∴ "+" contin.

For scalar multiplication, suppose $\alpha\in\mathbb{k}$, $x\in\mathcal{V}$ are s.t. $\alpha[x]\in U$. $\Rightarrow \alpha x\in\pi^{\leftarrow}U\in\mathcal{T}_{\mathcal{V}}$.

$\Rightarrow \exists \varepsilon>0,\ V_x \in \mathcal{T}_N$ s.t. $x \in V_x$ and $\beta V_x \subset \pi^{\leftarrow} U \quad \forall |\beta-\alpha|<\varepsilon$.
$\Rightarrow \pi(V_x) \in \mathcal{T}_{N/N}$, $[x] \in \pi(V_x)$ and $\pi(\beta V_x) = \beta\pi(V_x) \subset U$ $\forall |\beta-\alpha|<\varepsilon$. This proves continuity of scalar multip.
$\therefore$ N/N is a topol. vect space and π is contin. and open. ■

"(c)" $0 \in U \in \mathcal{T}_{N/N} \overset{\pi(0)=0}{\Rightarrow} 0 \in \pi^{\leftarrow} U \in \mathcal{T}_N \Rightarrow \exists V \in \mathcal{B}$ s.t. $0 \in V \subset \pi^{\leftarrow} U$
$\Rightarrow 0 \in \pi(V) \subset U$. ■

"(d)" If $x' = x + n_1$, $y' = y + n_2$ with $n_1, n_2 \in N$, and $z \in N$, then $d(x'-y', z) = d(x-y+n_1-n_2, z)$
$= d(x-y, \underbrace{z-n_1+n_2}_{\in N})$. Thus the definition of $\tilde{d}$ does not depend on choice of representatives.
$d \geq 0 \Rightarrow \tilde{d} \geq 0$. If $[x] \neq [y] \Rightarrow x-y \notin N \underset{N \text{ closed}}{\Rightarrow} \exists \varepsilon>0$ s.t. $x-y+B(0,\varepsilon) \subset N^c$. Thus $z \in N$ implies $d(x-y, z) \geq \varepsilon \Rightarrow \tilde{d}([x],[y]) \geq \varepsilon > 0$. If $[x]=[y]$ $\Rightarrow x-y \in N \Rightarrow 0 = d(x-y, x-y) \geq \tilde{d}([x],[y]) \geq 0$.
Thus $\tilde{d}([x],[y])$ iff. $[x]=[y]$.
$\tilde{d}([y],[x]) = \inf\{\underbrace{d(y-x,z)}_{= d(0, z-y+x) = d(-z, x-y) = d(x-y, -z)} \mid z \in N\} \overset{-N=N}{=} \tilde{d}([x],[y])$.
Since $\forall z', z \in N$: $d(x-y, z) + d(y-y', z')$
$= d(x, z+y) + d(y+z, z'+y'+z)$
$\geq d(x, z'+z+y') = d(x-y', z'+z) \geq \tilde{d}([x],[y'])$
$\Rightarrow \tilde{d}([x],[y]) + \tilde{d}([y],[y']) \geq \tilde{d}([x],[y'])$.
Thus $\tilde{d}$ is a metric. It is obviously invariant, as $d(x-y, z) = d(x+y'-(y+y'), z)$.
By c), a local base for $\mathcal{T}_{N/N}$ is given by $\{\pi(B(0,r))\}_{r>0}$. If $x \in N$ has $d(x,0)<r$ then $\tilde{d}(x+N, 0) = \inf\limits_{z\in N} d(x-0, z) \leq d(x,0) < r$.

Also $\tilde{d}(x+N, 0) < r \Rightarrow \exists z_0 \in N$ s.t. $d(x, z_0) < r$
$\Rightarrow d(x-z_0, 0) < r$ and $\pi(x-z_0) = x+N$.
Thus $B_{\tilde{d}}(0,r) = \pi(B(0,r))$, which implies that $\mathcal{T}_{N/N}$ is induced by $\tilde{d}$. ■

(e) Since "$\tilde{d}$" corresponding to $d(x,y) = \|x-y\|$ satisfies $\tilde{d}(\pi(x),\pi(y)) = \inf_{z\in N} \|x-y-z\| = \|\pi(x-y)\|_{V/N}$ $= \|\pi(x)-\pi(y)\|_{V/N}$, it suffices to show that $\|\cdot\|_{V/N}$ is a norm. For this consider $\alpha\in\mathbb{K}$, $\alpha\neq 0$, and $x,y\in V$.
$\|\alpha\pi(x)\| = \|\pi(\alpha x)\| = \inf_{z\in N}\|\alpha x - z\| = |\alpha| \inf_{z\in N} \|x-\frac{1}{\alpha}z\|$ $= |\alpha|\,\|\pi(x)\|$.
In addition, $0\pi(x) = \pi(0x) = \pi(0) = N \Rightarrow \|0\pi(x)\| = 0$.
Thus $\|\alpha\pi(x)\| = |\alpha|\|\pi(x)\|$ $\forall \alpha\in\mathbb{K}$.
If $[x]\neq 0 \Rightarrow 0 < \tilde{d}([x],0) = \|\pi(x)\|_{V/N}$; and for $x,y\in V$ $\|\pi(x)+\pi(y)\| = \|\pi(x)-\pi(-y)\| = \tilde{d}(\pi(x),\pi(-y))$ $\leq \tilde{d}(\pi(x),0) + \tilde{d}(0,-\pi(y)) = \|\pi(x)\| + \|\pi(y)\|$.
$\therefore$ $\|\cdot\|_{V/N}$ is a norm. ■

(f) V locally convex $\Rightarrow \exists$ convex local base $\mathcal{B}$
$\overset{c)}{\Rightarrow}$ $\{\pi(V)\}_{V\in\mathcal{B}}$ is a local base, and each $\pi(V)$ is convex by 5.1.b).

V localy bounded $\Rightarrow \exists V_0\in\mathcal{T}_V$ s.t. $0\in V_0$ and V_0 bounded
$\overset{b)}{\Rightarrow}$ $\pi(V_0)\in\mathcal{T}_{V/N}$ and $0\in\pi(V_0)$. Let $W\in\mathcal{T}_{V/N}$ s.t. $0\in W$
$\Rightarrow 0\in\pi^{\leftarrow}W\in\mathcal{T}_V \Rightarrow \exists s>0$ s.t. $V_0\subset t\pi^{\leftarrow}W$ $\forall t>s$.
But then $[x]\in\pi(V_0) \Rightarrow \exists x_0\in V_0$ s.t. $x_0\sim x$.
Now $\frac{1}{t}x_0\in\pi^{\leftarrow}W \Rightarrow \pi(\frac{1}{t}x_0) = \frac{1}{t}\pi(x_0)\in W$
$\Rightarrow \pi(x) = \pi(x_0)\in tW$. Thus $\pi(V_0)\subset tW$, which proves that $\pi(V_0)$ is bounded.

V metrizable $\overset{7.1.}{\Rightarrow}$ $\exists$ compatible invariant metric on V
$\overset{d)}{\Rightarrow}$ $\exists$ compatible inv. metric on V/N.

V normable $\overset{e)}{\Rightarrow}$ V/N normable

V F-space $\Rightarrow \exists$ compatible invariant metric d on V s.t. V is complete. We need to prove that then V/N is complete under $\tilde{d}$. Let $\{[y_n]\}_{n\in\mathbb{N}_+}$ be Cauchy sequence in V/N, $\Rightarrow \forall k\in\mathbb{N}_+$ $\exists n_k\in\mathbb{N}_+$ s.t. $\tilde{d}([y_{n'}],[y_n])<2^{-k}$ $\forall n',n\geq n_k$. Defining $m_k := \max_{1\leq \ell\leq k}(n_\ell, k)$ yields a subseq. s.t. $\tilde{d}([y_{m_k}],[y_{m_{k+1}}]) < 2^{-k}$ $\forall k\in\mathbb{N}_+$. Thus $\forall k\in\mathbb{N}_+$ $\exists z_k\in N$ s.t. $d(y_{m_k}-y_{m_{k+1}}, z_k) < 2^{-k}$. Define $\forall k\in\mathbb{N}_+$:
$x_k = y_{m_k} + \sum_{\ell=1}^{k-1} z_\ell \in V \Rightarrow d(x_k, x_{k+1}) < 2^{-k}$ $\forall k\in\mathbb{N}_+$

$\Rightarrow d(x_k, x_{k+m}) < 2^{1-k}$ $\forall m\geq 0$. Thus (x_k) is Cauchy in V

$\Rightarrow \exists x \in \mathcal{V}$ s.t. $\lim_{k\to\infty} x_k = x \overset{b)}{\Rightarrow} \lim_{k\to\infty} \pi(x_k) = \pi(x)$

But since $x_k - y_{m_k} \in N \Rightarrow \pi(x_k) = y_{m_k} + N$. Thus $([y_n])$ has a convergent subsequence and is Cauchy $\Rightarrow \lim_{n\to\infty} [y_n] = \pi(x)$. $\therefore \mathcal{V}/N$ is complete. ■

$\mathcal{V}$ Fréchet $\Rightarrow$ F-space and locally convex $\overset{\text{above}}{\Rightarrow}$ $\mathcal{V}/N$ F-space and locally convex. $\Rightarrow$ Fréchet.

$\mathcal{V}$ Banach $\overset{(e)}{\Rightarrow}$ $\mathcal{V}/N$ normable, with $\tilde{d}(\pi(x), \pi(y)) = \|\pi(x) - \pi(y)\|$. Above result shows that $\mathcal{V}/N$ is complete in $\tilde{d}$ $\Rightarrow$ $\mathcal{V}/N$ Banach. □

* In the exercises, it is also shown that if $\mathcal{H}$ = Hilbert and N a closed subspace, then $\exists$ scalar product on $\mathcal{H}/N$ s.t. $(\pi(x), \pi(x)) = \|\pi(x)\|^2_{\mathcal{H}/N}$, and that then $\mathcal{H}/N \cong N^{\perp}$ = closed subspace of $\mathcal{H}$. This shows that $\mathcal{H}/N$ is a Hilbert space, adding one more item to (f) above.

9.4. Theorem Let $\mathcal{V}$ be a topol. vect. space, and assume N and F are its subspaces. If N is closed and F has finite dimension, then $N+F$ is a closed subspace.

Proof. Consider the quotient map $\pi_N : \mathcal{V} \to \mathcal{V}/N$, and let $\mathcal{V}/N$ have the above quotient topology. Let $(e_n)_{n=1}^{d_F}$ denote a basis of F. Obviously, $\pi_N(F) = \operatorname{span}\left(\pi_N(e_n)\right)_{n=1}^{d_F} \Rightarrow \pi_N(F)$ is a finite dim. subspace of the topol. vect. space $\mathcal{V}/N \overset{6.1.}{\Rightarrow} \pi_N(F)$ is closed in $\mathcal{V}/N$. As $F+N = \pi_N^{\leftarrow}(\pi_N(F))$ and π_N is continuous and linear, the implies that $N+F$ is a closed subspace in $\mathcal{V}$. □ (5.1.c)

* If F is merely closed, it can happen that $N+F$ is not closed in $\mathcal{V}$.

10. The Banach-Steinhaus Theorem

10.1. Baire Category Theory:

Definition: Let $\mathfrak{X}$ be a set with topology $\mathcal{T}_{\mathfrak{X}}$.

* If $E \subset \mathfrak{X}$ is s.t. $\overline{E}$ has empty interior, E is called <u>nowhere dense</u>.

* If $E \subset \mathfrak{X}$ is s.t. there is a <u>countable</u> collection of <u>nowhere dense</u> sets $F_n \subset \mathfrak{X}$ for which $E = \bigcup F_n$, then E is of <u>the first category</u> in $\mathfrak{X}$. (these sets are also called <u>meager</u> or of the first Baire category.)

* If $E \subset \mathfrak{X}$ is not of the first category, it is of the <u>second category</u> (or nonmeager).

10.2. Proposition (basic properties)

(a) If $A \subset B \subset \mathfrak{X}$ and B is of the first category in $\mathfrak{X}$, then so is A.
(b) A countable union of sets of the 1st category is of the 1st category.
(c) If $E \subset \mathfrak{X}$ is closed and $E^\circ = \emptyset \Rightarrow E$ is of the 1st category
(d) If $\Phi : \mathfrak{X} \to \mathfrak{X}$ is a homeomorphism and $E \subset \mathfrak{X}$, then E and $\Phi(E)$ have the same category in $\mathfrak{X}$.

<u>Proof.</u> (a) $B = \bigcup_{n \in \mathbb{N}} F_n$, $A \subset B \Rightarrow A = \bigcup_{n \in \mathbb{N}} (A \cap F_n)$, and U open, $U \subset \overline{A \cap F_n} \subset \overline{F}_n \Rightarrow U = \emptyset$.
(b) Obvious ($\mathbb{N} \times \mathbb{N}$ is countable).
(c) Obvious.
(d) $E = \bigcup_n F_n \Rightarrow \Phi(E) = \bigcup_n \Phi(F_n)$. If U open and $U \subset \overline{\Phi(F_n)} \underset{\Phi \text{ homeo.}}{=} \Phi(\overline{F}_n) \Rightarrow \Phi^{-1}(U) \subset \overline{F}_n \Rightarrow \Phi^{-1}(U) = \emptyset$
$\Rightarrow U = \emptyset$ □

& $\Phi(E) = \bigcup_n F_n \underset{\Phi \text{ bij.}}{\Rightarrow} E = \bigcup_n \Phi^{-1}(F_n)$.

10.3. Baire's theorem:

If X is either ① a complete metric space or ② a locally compact Hausdorff space, then $\bigcap_{n\in I} V_n$ is dense in X, whenever I is countable and each $V_n \subset X$ is dense and open.

Proof: If I is finite, define $V_n = X$ $\forall n > |I|$. Since X is dense and open in X and then $\bigcap_{n\in\mathbb{N}_+} V_n = \bigcap_{n=1}^{\infty} V_n$, it suffices to prove the theorem for $I = \mathbb{N}_+$.

Consider then $W \in \mathcal{T}_X$ which is not empty. Then for any $n \in \mathbb{N}_+$ $W \cap V_n$ is open and non-empty (as $\overline{V_n} = X$). Choose $x_0 \in W \cap V_n$. Consider first case ①, i.e., X is complete metric space. Then $\exists \varepsilon > 0$ s.t. $\varepsilon < \frac{1}{n}$, $B(x_0, 2\varepsilon) \subset W \cap V_n$ $\Rightarrow$ $\overline{B(x_0,\varepsilon)} \subset B(x_0, 2\varepsilon) \subset W \cap V_n$.

Thus if $W_0 \in \mathcal{T}_X$, $W_0 \neq \emptyset$, we can iteratively find $x_n \in W_0$, $0 < \varepsilon_n < \frac{1}{n}$ $\forall n \in \mathbb{N}_+$, s.t. with $W_n := B(x_n, \varepsilon_n)$ we have $\overline{W_n} \subset W_{n-1} \cap V_n$. Let $K = \bigcap_{n\in\mathbb{N}_+} \overline{W_n}$. Clearly, $W_n \subset \bigcap_{k=0}^{n-1} W_k$ and (x_n) is a Cauchy sequence in X $\Rightarrow \exists\, \bar{x} = \lim_{n\to\infty} x_n$, and for any $n \in \mathbb{N}_+$, we have $\bar{x} \in \overline{\{x_k\}_{k\geq n}} \subset \overline{W_n}$

$\Rightarrow$ $\bar{x} \in K$ and thus $K \neq \emptyset$.

But obviously $K \subset W_0 \cap (\bigcap_{n\in\mathbb{N}_+} V_n)$. As W_0 was an arbitrary neighborhood, this proves that $\bigcap V_n$ is dense in X. (cf. Proof of Thrm 7.3.)

Assume then that X is a locally compact Hausdorff space. By this, we mean that every point has a neighborhood whose closure is compact. Then if $W \in \mathcal{T}_X$, $W \neq \emptyset$, and $x_0 \in W \cap V_1$, $\exists U \in \mathcal{T}_X$ s.t. $x_0 \in U$ and $\overline{U}$ is compact, and $\overline{U} \subset W \cap V_1$. (Rudin, RCA, Theorem 2.7. applied to $\{x_0\}$.) Thus for any $W_0 \in \mathcal{T}_X$, $W_0 \neq \emptyset$, we can iteratively choose $W_n \in \mathcal{T}_X$, s.t. $W_n \neq \emptyset$, $\overline{W_n}$ is compact, and $\overline{W_n} \subset V_n \cap W_{n-1}$ $\forall n \in \mathbb{N}_+$. Then $\bigcap_{n\in\mathbb{N}_+} \overline{W_n} \neq \emptyset$, by the same argument as used in the Proof of Lemma 6.2. $\Rightarrow W_0 \cap (\bigcap_n V_n) \neq \emptyset$ $\square$
(finite intersection property)