MATHEMATICS AND STATISTICS DEPARTMENT
INTRODUCTION TO STATISTICS FALL 2010
EXERCISE 5

1. At a certain stage of a criminal investigation the inspector in charge is 60 percent convinced of the guilt of a certain suspect. Suppose a new piece of evidence that shows that the criminal has a certain characteristic, such as left handedness, baldness, or brown hair is uncovered. If 20 percent of the population possesses this characteristic, how certain of the guilt of the suspect should the inspector now be if it turns out that the suspect has this characteristic?.

Let G = suspect is guilty. C = suspect has characteristic.

WANT: the improved probability of G given the characteristic, $P(G|C)$.
GIVEN: $P(G) = 0.6$ *and that 20% of the population has the characteristic.*
Suppose the suspect is NOT guilty (G^c) *but has the characteristic. Then we can assume the suspect is part of 20% of the population and write:* $P(C| G^c) = 0.2$

$P(G\cap C) = P(C|G)P(G)= 1*0.6$, *if suspect is truly guilty, then evidence says s/he must have characteristic!*
$P(C) = P(C\cap G) + P(C\cap G^c) = P(C|G)P(G) + P(C| G^c)P(G^c) = 1*0.6 + 0.2*(1-0.6)= 0.68$

Then using Bayes' theorem:
$P(G|C) = P(G\cap C) / P(C) = 0.6 / 0.68 = 0.882$

2. An insurance company believes that people can be divided into two classes: those who are accident prone and those who are not. Their statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability 0.4, whereas this probability decreases to 0.2 for non-accident-prone person. If we assume that 30 percent of the population is accident prone,
(a) What is the probability that a new policyholder will have an accident within a year of purchasing a policy?

Let A = accident-prone person. A^c *= non-accident-prone person.*

GIVEN: $P(\text{accident occurs} \mid A) = 0.4$, $P(\text{accident occurs} \mid A^c) = 0.2$, $P(A) = 0.3$

$$\begin{aligned} P(\text{accident occurs}) &= P(\text{accident occurs} \cap A) + P(\text{accident occurs} \cap A^c) \\ &= P(\text{accident occurs} \mid A)*P(A) + P(\text{accident occurs} \mid A^c)*P(A^c) \\ &= 0.4 * 0.3 + 0.2 * 0.7 = 0.26 \end{aligned}$$

(b) Suppose now that a new policyholder has an accident within a year of purchasing a policy. What is the probability that he or she is accident prone?

$$\begin{aligned} P(A \mid \text{accident occurs}) &= P(\text{accident occurs} \cap A) / P(\text{accident occurs}) \\ &= P(\text{accident occurs} \mid A) * P(A) / P(\text{accident occurs}) \\ &= 0.4 * 0.3 / 0.26 = 6/ 13 \text{ or } 0.461538 \end{aligned}$$

(c) What is the conditional probability that a new policyholder will have an accident in his or her second year of policy ownership, given that the policyholder has had an accident in the first year?

The chance of a person being truly accident-prone increases (is more likely) after an accident the 1^{st} year, so we should update P(A) and $P(A^c)$ to reflect that for the 2^{nd} year:

Let B = accident-prone person given an accident occurred 1^{st} year.

WANT: P(accident 2^{nd} year) = P(accident 2^{nd} year ∩ B) + P(accident 2^{nd} year ∩ B^c)

GIVEN: P(B) = P(A | accident 1^{st} year) = 6/13 ***from part b)!***
$P(B^c)$ = 1 – P(B) = P(A^c| accident 1^{st} year) = 7/13
Conditional probabilities of an accident occurring are <u>FIXED (unchanging)</u> yearly probabilities.
P(accident 1^{st} year| A) = P(accident 2^{nd} year | B) = 0.4 = 2/5
P(accident 1^{st} year | A^c) = P(accident 2^{nd} year | B^c) = 0.2 = 1/5

*P(accident 2^{nd} year) = P(accident 2^{nd} year | B)*P(B) + P(accident 2^{nd} year | B^c)* P(B^c)*
= (2/5)(6/13) + (1/5)*(7/13) = (2*6+1*7) / (5*13) =* ***19/65 or 0.2923***

<u>Alternate interpretation with same solution</u>
Let A = accident-prone, 1^{st} = accident occurs 1^{st} year and 2^{nd}= accident occurs 2^{nd} year.

WANT: P(accident occurs 2^{nd} year | accident occurs 1^{st} year) = P(2^{nd} ∩ 1^{st}) / P(1^{st})

GIVEN: P(accident occurs 1^{st} year) = P(1^{st}) = 0.26 ***from part a)!***
Conditional probabilities of an accident occurring are <u>FIXED (unchanging)</u> yearly probabilities. P(1^{st} | A) = P(2^{nd} | A) = 0.4, and P(1^{st} | A^c) = P(2^{nd} | A^c) = 0.2

P(2^{nd} ∩ 1^{st}) = P(accident in both years given person is A) + P(accident in both years given person is A^c)
= P(1^{st}|A)P(2^{nd}|A)P(A) + P(1^{st}|A)P(2^{nd}|A)P(A)
*= 0.4 * 0.4 * 0.3 + 0.2 * 0.2 * 0.7 = 0.076*

Then, P(accident occurs 2^{nd} year | accident occurs 1^{st} year) = 0.076 / 0.26 = ***19/65 or 0.2923***