(Supplement to paper)

Evolutionary branching and long - term coexistence of cycling predators

General

Prey dynamics in absence of predators:

$\frac{d}{dt} x = r\, x\, (1 - x/K)$

Without loss of generality (i.e., by scaling time and prey population density) we may assume that $r = 1$ and $K = 1$. Hence we write

$\frac{d}{dt} x = x\, (1 - x)$

Resident dynamics:

$\frac{d}{dt} x = x\, (1 - x) - \sum_i \frac{\beta\, x\, y_i}{1+\beta\, h_i\, x}$ (prey)

$\frac{d}{dt} y_i = \frac{\gamma_i\, \beta\, x\, y_i}{1+\beta\, h_i\, x} - \delta\, y_i$ (predator)

Without loss of generality (i.e., by scaling predator population density) we may further assume that $\beta = 1$. Hence we write

$\frac{d}{dt} x = x\, (1 - x) - \sum_i \frac{x\, y_i}{1+h_i\, x}$ (prey)

$\frac{d}{dt} y_i = \frac{\gamma_i\, x\, y_i}{1+h_i\, x} - \delta\, y_i$ (predator)

Invader dynamics:

$\frac{d}{dt} y_{\text{mut}} = \frac{\gamma_{\text{mut}}\, x\, y_{\text{mut}}}{1+h_{\text{mut}}\, x} - \delta\, y_{\text{mut}}$

Invasion fitness:

$\left\langle \frac{d}{dt} \text{Log}[y_{\text{mut}}] \right\rangle = \gamma_{\text{mut}} \left\langle \frac{x}{1+h_{\text{mut}}\, x} \right\rangle - \delta$

Resident selective neutrality:

$0 = \left\langle \frac{d}{dt} \text{Log}[y_{\text{res}}] \right\rangle = \gamma_{\text{res}} \left\langle \frac{x}{1+h_{\text{res}}\, x} \right\rangle - \delta \Longrightarrow \left\langle \frac{x}{1+h_{\text{res}}\, x} \right\rangle = \frac{\delta}{\gamma_{\text{res}}}$

Monomorphic resident population dynamics

Resident dynamics :

In[1]:=
$$\text{dx[t] := x[t] (1 - x[t])} - \frac{\text{x[t] y[t]}}{\text{1 + hRes x[t]}};$$
$$\text{dy[t] :=} \frac{\gamma\text{Res x[t] y[t]}}{\text{1 + hRes x[t]}} - \delta\text{ y[t]};$$

Resident equilibrium :

In[4]:=
```
Solve[{dx[t] == 0, dy[t] == 0}, {x[t], y[t]}];
Simplify[%]
```

Out[4]=
$$\left\{\{x[t] \to 0,\ y[t] \to 0\},\ \{y[t] \to 0,\ x[t] \to 1\},\ \left\{y[t] \to \frac{\gamma\text{Res} \, (\gamma\text{Res} - (1 + \text{hRes})\,\delta)}{(\gamma\text{Res} - \text{hRes}\,\delta)^2},\ x[t] \to \frac{\delta}{\gamma\text{Res} - \text{hRes}\,\delta}\right\}\right\}$$

In[5]:=
$$\text{xEq :=} \frac{\delta}{\gamma\text{Res} - \text{hRes}\ \delta};$$
$$\text{yEq :=} \frac{\gamma\text{Res}\ (\gamma\text{Res} - (1 + \text{hRes})\ \delta)}{(\gamma\text{Res} - \text{hRes}\ \delta)^2};$$

Zero cline of the prey:

```
In[7]:= Solve[dx[t] == 0, y[t]];
        Simplify[%]
Out[8]= {{y[t] → -(-1 + x[t]) (1 + hRes x[t])}}
```

X-xoordinate of top of zero cline of the prey :

```
In[9]:= Solve[0 == D[-(-1 + x) (1 + hRes x), x], x]
```

Out[9]= $\left\{\left\{x \to \frac{-1 + \text{hRes}}{2\,\text{hRes}}\right\}\right\}$

In[10]:= $\mathbf{xTop := \frac{-1 + hRes}{2\,hRes};}$

Stability boundary in the (γ,h)-plane :

```
In[11]:= Solve[xTop == xEq, γRes];
         Simplify[%]
```

Out[12]= $\left\{\left\{\gamma\text{Res} \to \frac{\text{hRes}\,(1 + \text{hRes})\,\delta}{-1 + \text{hRes}}\right\}\right\}$

Predator extinction boundary in the (γ,h)-plane :

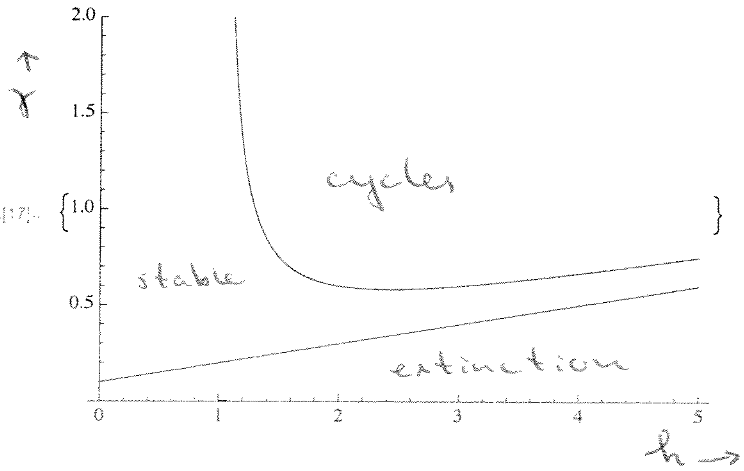
```
In[13]:= Solve[1 == xEq, γRes];
         Simplify[%]
Out[14]= {{γRes → (1 + hRes) δ}}
```

Bifurcation plot :

In[15]:= $\mathbf{stabPlot = Plot\left[\frac{hRes\,(1 + hRes)\,\delta}{-1 + hRes} \;/.\; \{\delta \to 0.1\},\ \{hRes,\ 1,\ 5\},\ PlotRange \to \{0,\ 2\}\right];}$

```
extPlot = Plot[(1 + hRes) δ /. {δ → 0.1}, {hRes, 0, 5}, PlotRange → {0, 2}];

{Show[extPlot, stabPlot]}
```

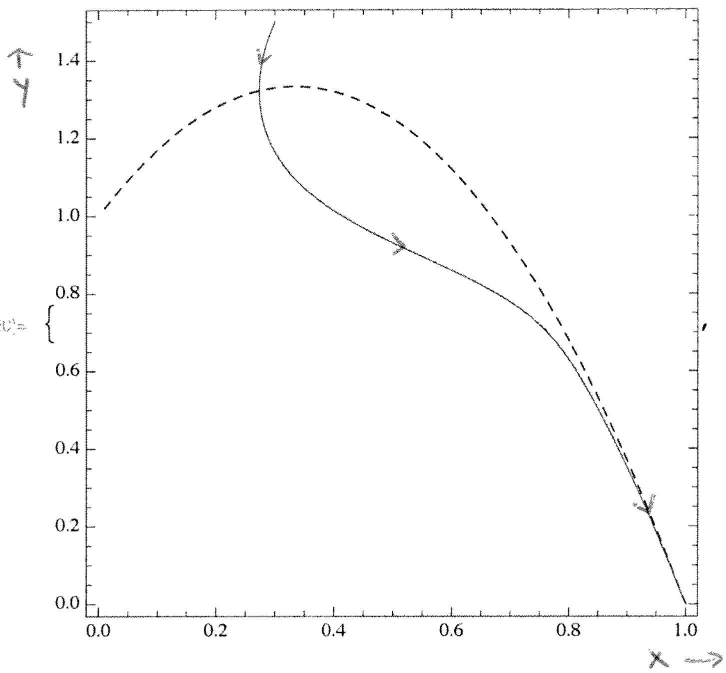
Out[17]= {}

In[18]:=
```
hRes = 3; γRes = 0.25; δ = 0.1; tMax = 100;

orbit =
  NDSolve[{x'[t] == dx[t], y'[t] == dy[t], x[0] == 0.3, y[0] == 1.5}, {x, y}, {t, 0, tMax}];

{
 Show[
  ContourPlot[dx[t] dy[t], {x[t], 0, 1}, {y[t], 0, 1.5}, Contours → {0},
   ContourShading → False, ContourStyle → {Dashed, Black}, PlotPoints → 50],
  ParametricPlot[Evaluate[{x[t], y[t]} /. orbit], {t, 0, tMax},
   AxesOrigin → {0, 0}, PlotRange → All]
 ],
 Plot[Evaluate[{x[t], y[t]} /. orbit], {t, 0, tMax}, AxesOrigin → {0, 0}]
}
```

Out[20]= {



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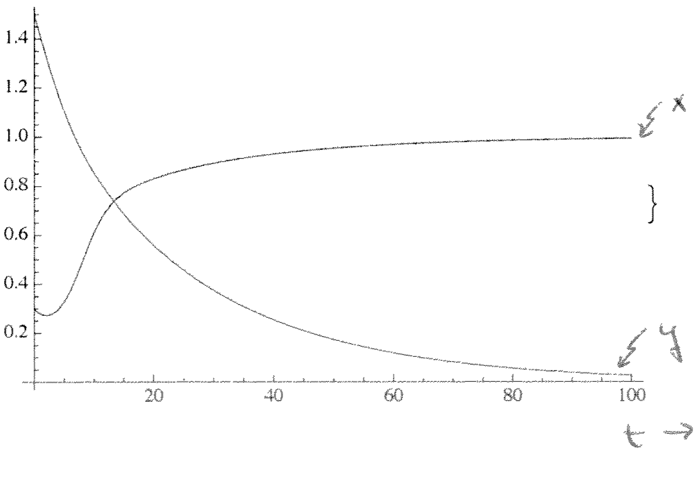
extinction of predator



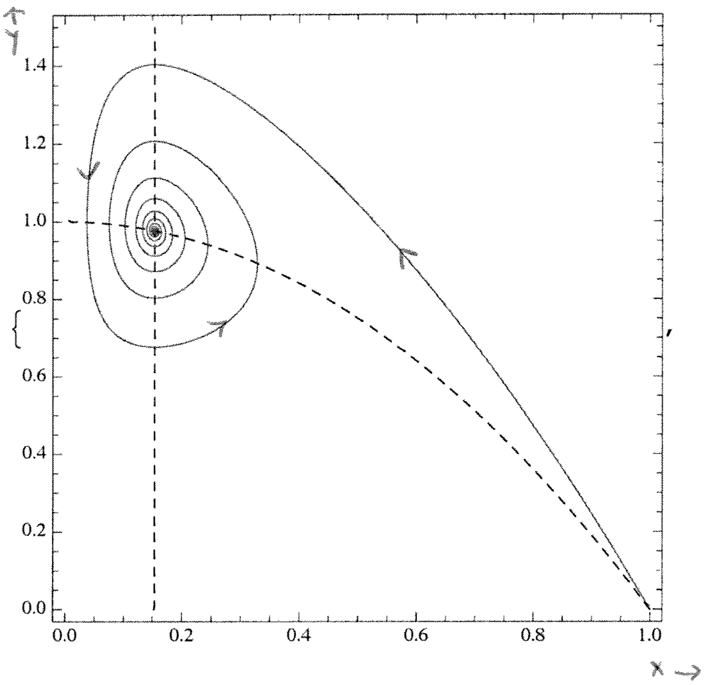
}

In[21]:=
```
hRes = 1; γRes = 0.75; δ = 0.1; tMax = 500;

orbit = NDSolve[{x'[t] == dx[t], y'[t] == dy[t], x[0] == 1, y[0] == 0.01}, {x, y}, {t, 0, tMax}];

{
 Show[
  ContourPlot[dx[t] dy[t], {x[t], 0, 1}, {y[t], 0, 1.5}, Contours → {0},
   ContourShading → False, ContourStyle → {Dashed, Black}, PlotPoints → 50],
  ParametricPlot[Evaluate[{x[t], y[t]} /. orbit], {t, 0, tMax},
   AxesOrigin → {0, 0}, PlotRange → All]
 ],
 Plot[Evaluate[{x[t], y[t]} /. orbit], {t, 0, tMax}, AxesOrigin → {0, 0}]
}
```

Out[23]= {



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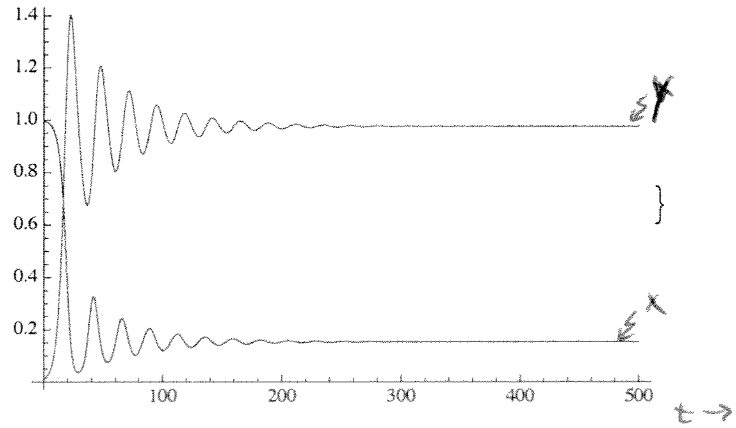
stable coexistence



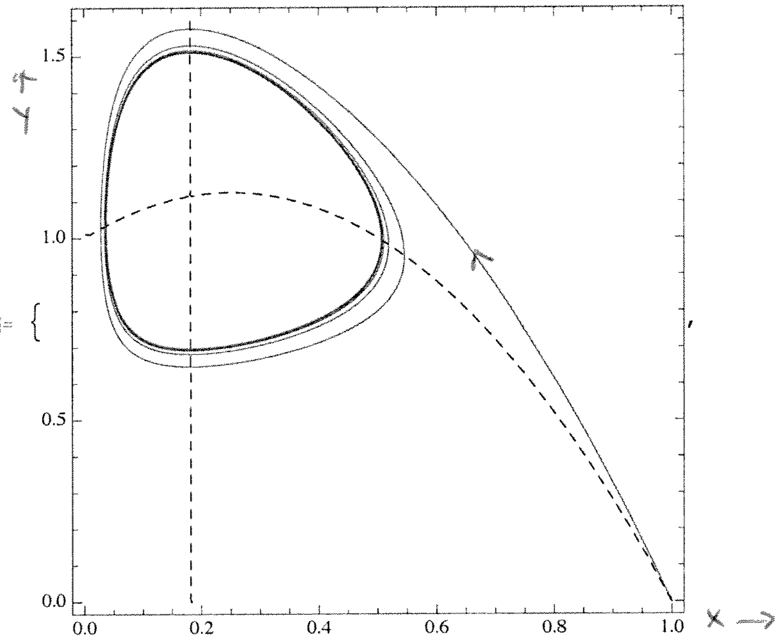
}

In[24]:=
```
hRes = 2; γRes = 0.75; δ = 0.1; tMax = 500;

orbit = NDSolve[{x'[t] == dx[t], y'[t] == dy[t], x[0] == 1, y[0] == 0.01}, {x, y}, {t, 0, tMax}];

{
 Show[
  ContourPlot[dx[t] dy[t], {x[t], 0, 1}, {y[t], 0, 1.6}, Contours → {0},
   ContourShading → False, ContourStyle → {Dashed, Black}, PlotPoints → 50],
  ParametricPlot[Evaluate[{x[t], y[t]} /. orbit], {t, 0, tMax},
   AxesOrigin → {0, 0}, PlotRange → All]
 ],
 Plot[Evaluate[{x[t], y[t]} /. orbit], {t, 0, tMax}, AxesOrigin → {0, 0}]
}
```

Out[26]= {



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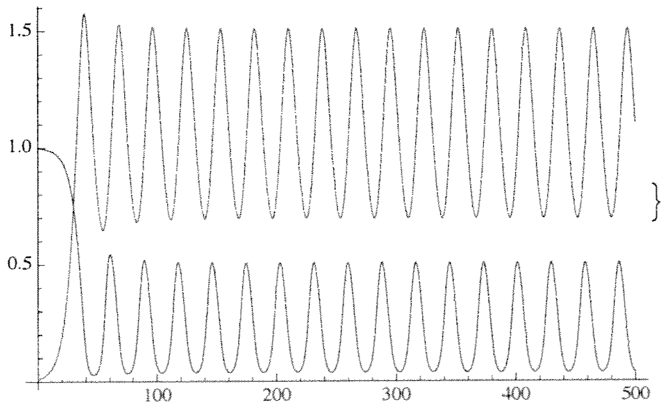
limit cycle



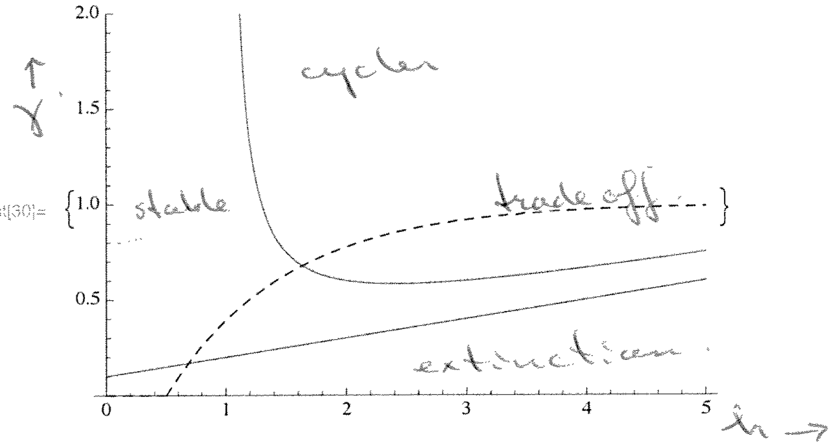
}

Trade - off :

In[27]:=
```
γ[h_] := Max[a (1 - e^(-b (h-hMin))), 0]
```

In[28]:=
```
a = 1; b = 1; hMin = 0.5;
tradeoff = Plot[γ[hRes], {hRes, 0, 5}, PlotRange → {0, 2}, PlotStyle → {Black, Dashed}];
{Show[extPlot, stabPlot, tradeoff]}
```

Out[30]=



Invasion in a stable monomorphic resident population

In[31]:=
```
Clear[hRes, γRes, δ]
```

Invasion fitness :

In[32]:=
$$s1 := \frac{\gamma Mut\ xEq}{1 + hMut\ xEq} - \delta;$$

Hence, $s1 > 0 \Longleftrightarrow xEq > \frac{\delta}{\gamma[hMut] - hMut\,\delta}$, i.e., selection minimizes the prey density at equilibrium.

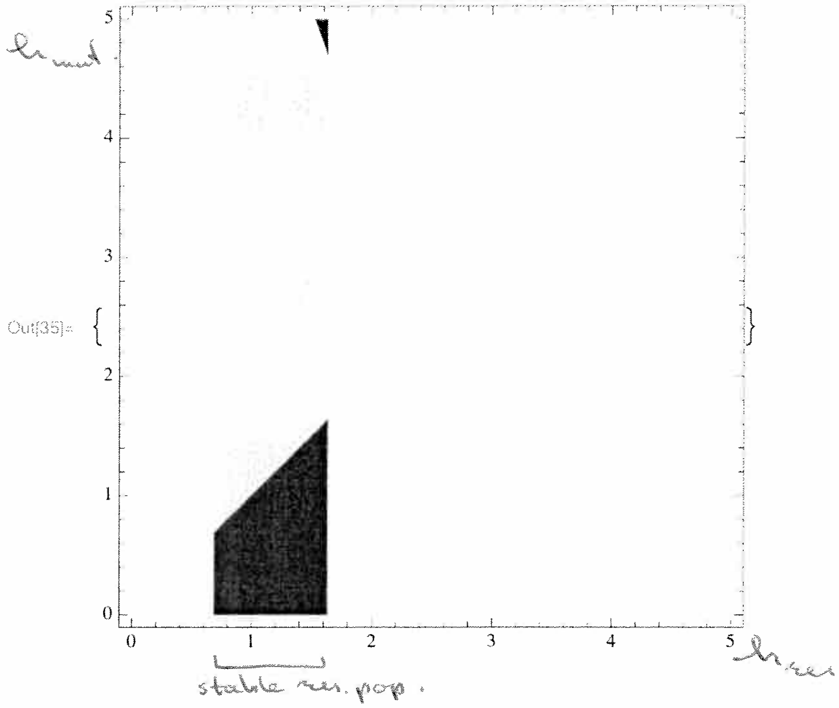
Pairwise invadability plot :

In[33]:=
```
arg = {γRes → γ[hRes], γMut → γ[hMut]};
```

```
In[34]:= a = 1; b = 1; hMin = 0.5; δ = 0.1;

{ContourPlot[If[(xTop /. arg) < (xEq /. arg) < 1, s1 /. arg],
  {hRes, 0, 5}, {hMut, 0, 5}, Contours → {0}]}
```

Out[35]=



- **Invasion in a cycling monomorphic resident population**

```
In[36]:= Clear[hRes, γRes, δ]
```

Invader dynamics :

```
In[37]:= dLogMut[t] :=  γMut x[t]
                       ───────────── - δ;
                       1 + hMut x[t]
```

Invasion fitness :

```
In[113]:= s1 := Block[{}, t0 = 500; tMax = 1000; γRes = γ[hRes]; γMut = γ[hMut];
    orbit =
     NDSolve[{x'[t] == dx[t], y'[t] == dy[t], x[0] == 1, y[0] == 0.01}, {x, y}, {t, 0, tMax}];
    NIntegrate[Evaluate[(   γMut x[t]
                         ───────────── - δ) /. orbit],
                         1 + hMut x[t]
      {t, t0, tMax}, MaxRecursion → 100] / (tMax - t0)];
```

```
In[114]:= a = 1; b = 1; hMin = 0.5; δ = 0.1;
```

```
In[115]:= hRes = 2;
hMut = 2;
γRes = γ[hRes];
γMut = γ[hMut];
s1
```

```
Out[119]= {0.000566896}
```

Pairwise invadability plot :

In[122]:= `{ContourPlot[s1, {hRes, 0, 5}, {hMut, 0, 5}, Contours → {0}]}`

Out[122]=