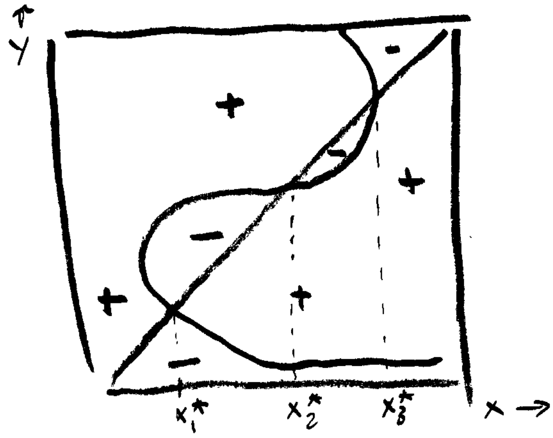
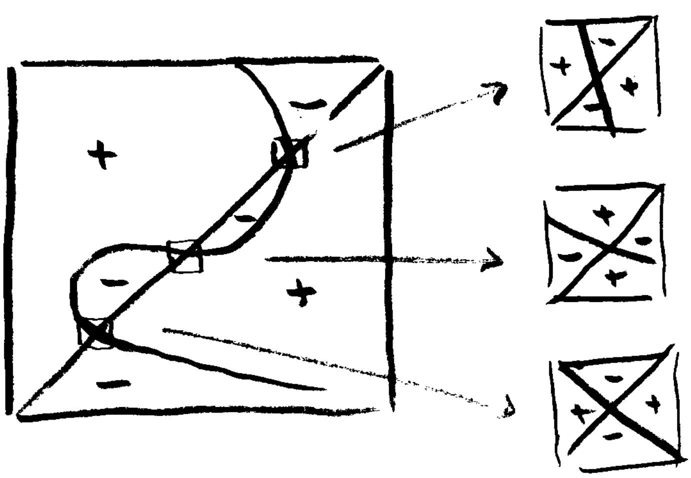
IV Classification of the local configuration of the PIP

Take any PIP:



We now know that with small mutation steps and away from the singular strategies x_1^*, x_2^* and x_3^* the successfully invading mutant will also replace the resident (i.e., invasion & substitution).

So, the interesting things happen near the singularities.
That is why we study the local configurations of the PIP around the singularities.



- Local configs in this PIP.
- Curved lines approximated by straight ones.

<u>Second-order Taylor approximation of $S_x(y)$ near a singular strategy x^*:</u>

$x^* \in \mathbb{R}^1$ singular, $\boxed{x, y \in \mathcal{U}_\varepsilon(x^*)}$, $\varepsilon > 0$.

① $$S_x(y) = [S_x(y)]^* + [\partial_x S_x(y)]^*(x-x^*) + [\partial_y S_x(y)]^*(y-x^*) +$$
$$+ \tfrac{1}{2}[\partial_{xx} S_x(y)]^*(x-x^*)^2 + [\partial_{xy} S_x(y)]^*(x-x^*)(y-x^*) +$$
$$+ \tfrac{1}{2}[\partial_{yy} S_x(y)]^*(y-x^*)^2 + O(\varepsilon^3)$$

where $[\ldots]^*$ means "evaluated for $x = x^*$ and $y = x^*$".

(i) $[S_x(y)]^* = 0$ because of the "selective neutrality of residents among themselves".

(ii) $[\partial_y S_x(y)]^* = 0$ because x^* is singular.

<u>What about the others?</u>

We have

② $S_x(x) = 0 \quad \forall x$ (selective neutrality of residents ...)

Differentiation of ② to x gives

(3) $$\left[\partial_x S_x(y)\right]_{y=x} + \left[\partial_y S_x(y)\right]_{y=x} = 0 \qquad \forall x$$

Substitute $x = x^*$, we find

(iii) $$\left[\partial_x S_x(y)\right]^* = -\left[\partial_y S_x(y)\right]^* \underset{(ii)}{=} 0$$

Differentiation of (3) to x gives

(4) $$\left[\partial_{xx} S_x(y)\right]_{y=x} + 2\left[\partial_{xy} S_x(y)\right]_{y=x} + \left[\partial_{yy} S_x(y)\right]_{y=x} = 0 \quad \forall x.$$

Substitute $x = x^*$, we find

(iv) $$\left[\partial_{xy} S_x(y)\right]^* = -\frac{1}{2}\left(\left[\partial_{xx} S_x(y)\right]^* + \left[\partial_{yy} S_x(y)\right]\right)^*$$

Substituting (i),...,(iv) into (1) and simplifying the expression, we get

(5) $$S_x(y) = \frac{1}{2}(x-y)\Big(C_{11}(x-x^*) - C_{22}(y-x^*)\Big) + O(\varepsilon^3)$$

where

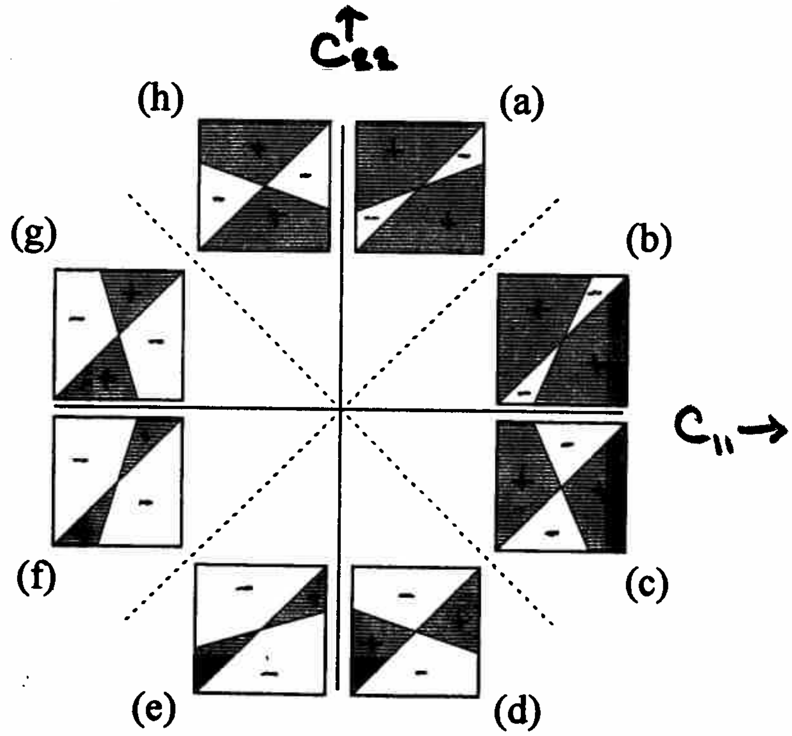
(6) $$\begin{cases} C_{11} := \left[\partial_{xx} S_x(y)\right]^* \\ C_{12} := \left[\partial_{xy} S_x(y)\right]^* \\ C_{22} := \left[\partial_{yy} S_x(y)\right]^*. \end{cases}$$

Ignoring the $O(\varepsilon^3)$ terms, we use ⑤ to make the local configuration of the PIP around x^*.

It can be seen from ⑤ that the local config. only depends on C_{11} and C_{22}.

So, we can classify the local configs. in terms of C_{11} and C_{22}:

<u>The "eight cases"</u> (Geritz et al 1998)



(N.B: <u>dark</u> means $S_x(y) > 0$
<u>light</u> means $S_x(y) < 0$)

From the previous figure we conclude:

$C_{22} < 0$: x^* is uninvadable (ESS) (evol. fixed point).

$C_{22} > 0$: x^* can be invaded by any nearby mutant.

$C_{11} < 0$: x^* is not a good invader itself.

$C_{22} > 0$: x^* can invade any nearby resident strategy

$C_{22} < C_{11}$: x^* is monomorphically attracting.

$C_{22} > C_{11}$: x^* is monomorphically repelling.

$C_{22} < -C_{11}$: there are no strategies near x^* that can coexist

$C_{22} > -C_{11}$: in every neighborhood of x^* there are strategies that can coexist.

If $c_{22} > -c_{11}$, then there are strategies x_1, x_2 near x^* that can coexist.

Who can invade such a dimorphic resident population?

Second-order Taylor approximation of $S_{x_1,x_2}(y)$ near a singular x^*.

$x^* \in \mathbb{R}^1$ singular, $|x_1, x_2, y \in U_\varepsilon(x^*)|$, $\varepsilon > 0$

(7) $$S_{x_1,x_2}(y) = \alpha + \beta_0(y-x^*) + \beta_1(x_1-x^*) + \beta_2(x_2-x^*) + \frac{1}{2}\gamma_{00}(y-x^*)^2 + \frac{1}{2}\gamma_{11}(x_1-x^*)^2 + \frac{1}{2}\gamma_{22}(x_2-x^*)^2 + \gamma_{01}(y-x^*)(x_1-x^*) + \gamma_{02}(y-x^*)(x_2-x^*) + \gamma_{12}(x_1-x^*)(x_2-x^*) + O(\varepsilon^3)$$

for given constants $\alpha, \ldots, \gamma_{12} \in \mathbb{R}$.
To find out what these constants actually are, we use that

(8) $$\begin{cases} S_{x_1,x_2}(x_1) = 0 & \forall x_1, x_2 \\ S_{x_1,x_2}(x_2) = 0 & \forall x_1, x_2. \\ S_{x^*,x^*}(y) = S_{x^*}(y) & \forall y. \end{cases}$$

$$\boxed{S_{x_1,x_2}(x_1) = 0 \quad \forall x_1, x_2} \Longrightarrow$$

$$\text{(i)} \begin{cases} \alpha = 0, \quad \beta_0 + \beta_1 = 0, \quad \beta_2 = 0, \\ \gamma_{00} + \gamma_{01} + \gamma_{11} = 0, \quad \gamma_{02} + \gamma_{12} = 0, \quad \gamma_{22} = 0 \end{cases}$$

$$\boxed{S_{x_1,x_2}(x_2) = 0 \quad \forall x_1, x_2} \Longrightarrow$$

$$\text{(ii)} \begin{cases} \alpha = 0, \quad \beta_0 + \beta_2 = 0, \quad \beta_1 = 0 \\ \gamma_{00} + \gamma_{02} + \gamma_{22} = 0, \quad \gamma_{01} + \gamma_{12} = 0, \quad \gamma_{11} = 0 \end{cases}$$

$$\left.\begin{aligned} &\boxed{S_{x^*,x^*}(y) = S_{x^*}(y) \quad \forall y} \\ &S_{x^*,x^*}(y) \overset{7}{=} \alpha + \beta_0 (y - x^*) + \gamma_{00} (y - x^*)^2 \\ &S_{x^*}(y) \overset{5}{=} \tfrac{1}{2} C_{22} (y - x^*)^2 \end{aligned}\right\} \Rightarrow$$

$$\text{(iii)} \quad \alpha = 0, \quad \beta_0 = 0, \quad \gamma_{00} = \tfrac{1}{2} C_{22}$$

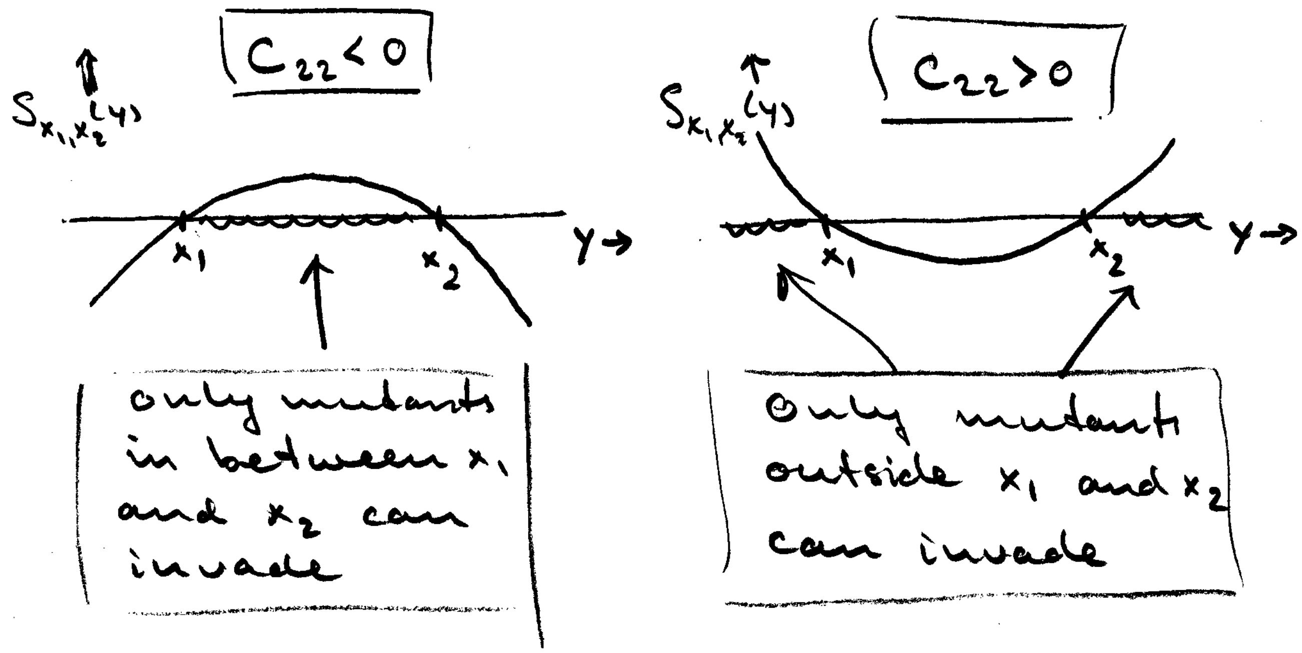
Combining (i), (ii) and (iii) we get:

$$\text{(iv)} \begin{cases} \alpha = \beta_0 = \beta_1 = \beta_2 = \gamma_{11} = \gamma_{22} = 0 \\ \gamma_{00} = \gamma_{12} = \tfrac{1}{2} C_{22} \\ \gamma_{01} = \gamma_{02} = -\tfrac{1}{2} C_{22} \end{cases}$$

Substitute (iv) into (7) and simplify:

(9) $$S_{x_1,x_2}(y) = \frac{1}{2} C_{22} (y-x_1)(y-x_2) + O(\varepsilon^3)$$

$\Rightarrow$ <u>Fitness landscapes:</u>



If $\boxed{C_{22}\neq 0}$ then three similar strategies cannot coexist (see Prop. (1) some lectures ago).

So, after invasion one of the residents or both, will be eliminated.

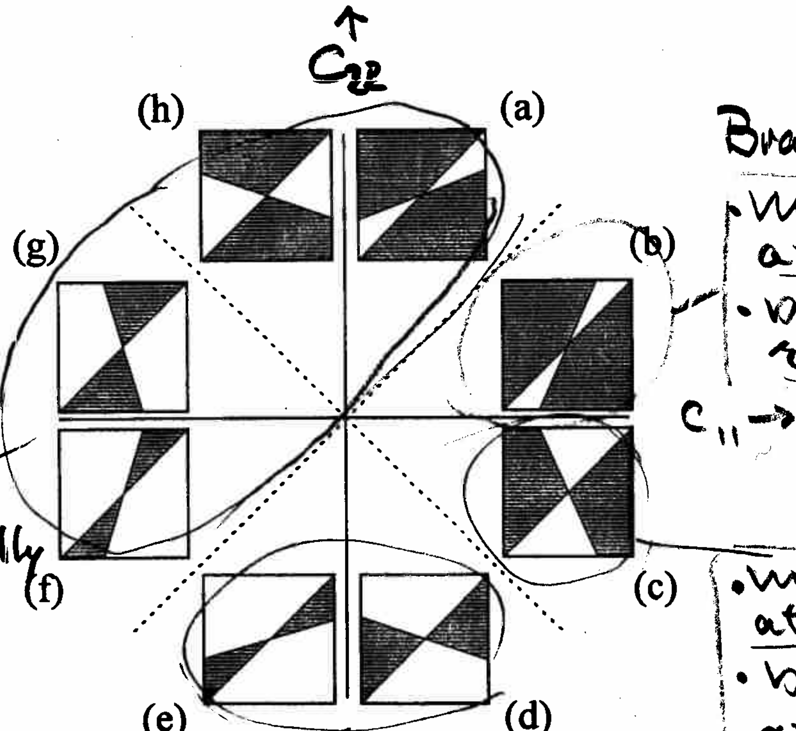
If $\boxed{C_{22}<0}$ then $x_1 < y < x_2$ and one or both of the extreme strategies (x_1, x_2) will be eliminated.

$\Rightarrow$ $\boxed{x^* \text{ is } \underline{\text{dimorphically}}\ \underline{\text{attracting}}}$.

If $\boxed{C_{22} > 0}$ then $y < x_1 < x_2$ or $x_1 < x_2 < y$, and the middle strategy will be eliminated while the population stays dimorphic.

$\Rightarrow$ $\boxed{x^* \text{ is } \underline{\text{dimorphically repelling}}}$

Conclusion



Branching Point:
- Monomorphically attracting
- Dimorphically <u>repelling</u>

Monomorphically repelling

- Monomorphically attracting
- Dimorphically <u>attracting</u>

- monomorphically attracting
- no coexistence nearby