IV Comparison of different notions of stability.

We already introduced the notion of <u>total stability</u>:

A singular point $\underline{x}^* := (x_1^*, \dots, x_k^*)$ is totally stable if it is stable for the differential inclusion

(1) $$0 \leq S'_{\underline{x}}(x_i)\,\dot{x}_i \quad (i = 1, \dots, k)$$

A sufficient condition for total stability is that there exist $d_1, \dots, d_k > 0$ such that

(2) $$\frac{a_{ii}}{d_i} < -\sum_{j \neq i} \frac{|a_{ij}|}{d_j} \quad \forall i.$$

where

(3) $$a_{ij} := \left[\partial_{yy} S_{\underline{x}}(y)\,\delta_{ij} + \partial_{x_j y} S_{\underline{x}}(y)\right]_{\substack{y = x_i^* \\ \underline{x} = \underline{x}^*}}$$

where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

is the Kronecker delta.

$A := (a_{ij})_{i,j=1}^k$ is the Jacobi-matrix of the vector field $\left(\partial_y S_{\underline{x}^*}(y)\big|_{y=x_i^*}\right)_{i=1}^k$

For the <u>canonical equation</u>

(4) $$\boxed{\dot{x}_i = \frac{1}{2}\mu(x_i)\sigma^2(x_i)\, n_E(x_i)\, S'_E(x_i) \qquad (i=1,\dots,k)}$$

we introduce two kinds of stability.

To that end we linearize (4) around the singular point $\underline{x}^* = (x_1^*, \dots, x_k^*)$:

(5) $$\boxed{(x_i - x_i^*)^{\cdot} = \left[\frac{1}{2}\mu(x_i^*)\sigma^2(x_i^*)\, n_E(x_i)\right] \sum_{j=1}^{k} a_{ij}(x_j - x_j^*)}$$

for all i, where the a_{ij} are the same as in (3) on the prev. page.

<u>Write:</u>

$A := (a_{ij})$ (<u>matrix</u>)

$C :=$ diagonal $\left(\frac{1}{2}\mu(x_i^*)\sigma^2(x_i^*)\, n_E(x_i^*)\right)$ (<u>matrix</u>)

Then (5) becomes

(6) $$\boxed{(\underline{x} - \underline{x}^*)^{\cdot} = C A (x - \underline{x}^*)}$$

Definition.

- $\underline{x}^*$ is **weakly stable** for given strictly positive diagonal matrix C_0 if x^* is stable in (6) with $C = C_0$.

- $\underline{x}^*$ is **strongly stable** if x^* is stable in (6) for every strictly positive diagonal matrix C.

∎

$$\boxed{\text{Special case } k=1}$$

$$(x - x^*)^{\cdot} = c\, a_{11} (x - x^*)$$

Stable if $a_{11} < 0$ and unstable if $a_{11} > 0$.

This does not depend on $c > 0$.

$\Rightarrow$ For $k=1$, strong and weak stability are equivalent.
Moreover, both are equivalent to total stability as well.

Special case : $k=2$

$$(\underline{x}-\underline{x}^*)^{\cdot} = CA(x-\underline{x}^*)$$

with

$$CA = \begin{pmatrix} c_1 a_{11} & c_1 a_{12} \\ c_2 a_{21} & c_2 a_{22} \end{pmatrix}$$

determinant   trace

$$\text{Stability} \iff \boxed{\det CA > 0} \;\&\; \boxed{\mathrm{tr}\, CA < 0}$$

(Routh - Hurwitz stability criterion)

Weakstability (for given $C>0$).

$$\det CA > 0 \iff a_{11} a_{22} > a_{12} a_{21}$$

$$\mathrm{tr}\, CA < 0 \iff c_1 a_{11} + c_2 a_{22} < 0 .$$

Strong stability (for any $C>0$)

$$\det CA > 0 \;\; \forall C > 0 \iff a_{11} a_{21} > a_{12} a_{21}$$

$$\mathrm{tr}\, CA < 0 \;\; \forall C>0 \iff c_1 a_{11} + c_2 a_{22} < 0 \;\; \forall C>0$$

$$\iff a_{11} < 0 \;\&\; a_{22} < 0 .$$

Summary

Total stability:

$a_{11} < 0$

$a_{22} < 0$

$a_{11} a_{22} > |a_{12} a_{21}|$

(from differential inclusion)

$$\Downarrow$$

Strong stability:

$a_{11} < 0$

$a_{22} < 0$

$a_{11} a_{22} > a_{12} a_{21}$

(from canonical equation for any $c > 0$)

$$\Downarrow$$

Weak stability

$c_1 a_{11} + c_{22} a_{22} < 0$

$a_{11} a_{22} > a_{12} a_{21}$

(from canonical equation for given $c > 0$)