Adaptive dynamics

Exercise 1

$$s_x(y) = r(y)\left(1 - \frac{a(y,x)\,K(x)}{K(y)}\right) = 0 \qquad , r(y) > 0 \ \forall y$$

$$\Rightarrow \quad 1 - \frac{a(y,x)\,K(x)}{K(y)} = 0 \Rightarrow \boxed{K(y) = a(y,x)\,K(x)}$$

a) $$\begin{cases} \dot{n} = \dfrac{dn}{dt} = r(x)n\left(1 - \dfrac{n + a(x,y)m}{K(x)}\right) \\ \dot{m} = \dfrac{dm}{dt} = r(y)m\left(1 - \dfrac{m + a(y,x)n}{K(y)}\right) \end{cases}$$

$$\underline{\dot{n} = 0} \Rightarrow r(x)n\left(1 - \frac{n + a(x,y)m}{K(x)}\right) = 0$$

$$\Rightarrow \Big(\underset{r(x)>0\ \forall x!}{r(x) = 0 \text{ or}}\Big) \ \boxed{n = 0} \text{ or } \boxed{m = \frac{K(x) - n}{a(x,y)}}$$

$$\underline{\dot{m} = 0} \Rightarrow \boxed{m = 0} \text{ or } \boxed{m = K(y) - a(y,x)n}$$

If $n=0$, then

$$\dot{m} = 0 \Rightarrow m = 0 \quad \text{or} \quad m = K(y)$$

$$\dot{n} = 0 \Rightarrow m = 0 \quad \text{or} \quad m = \frac{K(x)}{a(x,y)}$$

If $m=0$, then

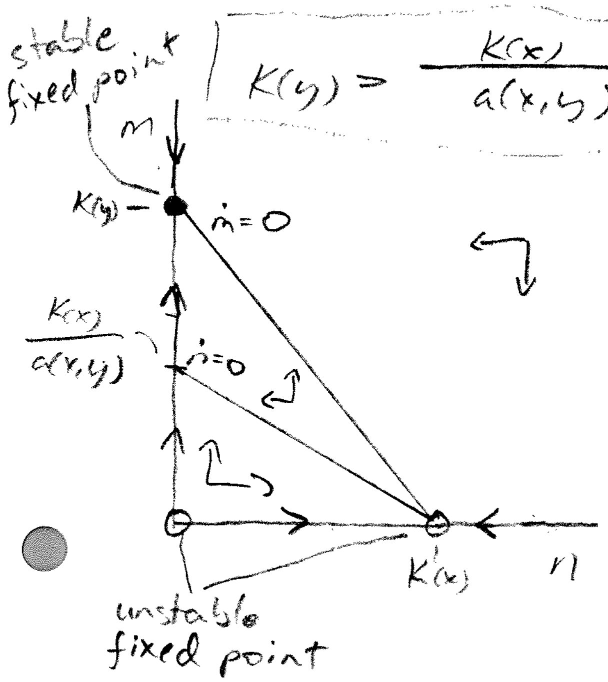
$$\dot{m} = 0 \Rightarrow m = 0 \quad \text{or} \quad n = \frac{K(y)}{a(y,x)}$$

$$\dot{n} = 0 \Rightarrow m = 0 \quad \text{or} \quad n = K(x)$$

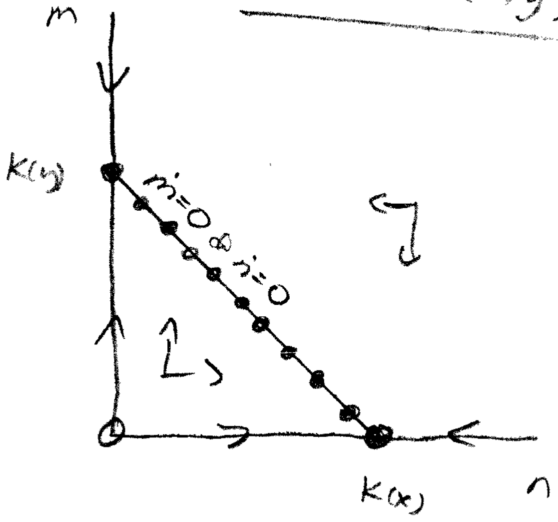
Note! $s_x(y) = 0 \Rightarrow K(x) = \dfrac{K(y)}{a(y,x)}$

We have three different cases:

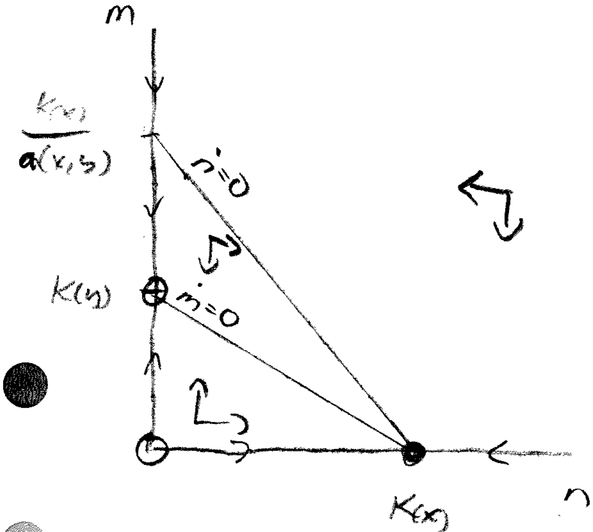
$$\boxed{K(y) > \frac{K(x)}{a(x,y)}}$$



$$\boxed{K(y) = \frac{K(x)}{a(x,y)}}$$



$$\boxed{K(y) < \frac{K(x)}{a(x,y)}}$$



b) Note! $\boxed{K(y) > \frac{K(x)}{a(x,y)}} \Rightarrow \frac{a(x,y)K(y)}{K(x)} > 1$

$\Rightarrow 1 - \frac{a(x,y)K(y)}{K(x)} < 0 \overset{r(x)>0\ \forall x}{\Rightarrow} r(x)\left(1 - \frac{a(x,y)K(y)}{K(x)}\right) < 0$

$\Rightarrow \boxed{S_y(x) < 0}$ (similarly $K(y) = \frac{K(x)}{a(x,y)} \Rightarrow S_y(x) = 0$ and $K(y) < \frac{K(x)}{a(x,y)} \Rightarrow S_y(x) < 0$)

From this and the phase-plane analysis we get:

i) $S_y(x) > 0$ ii) $S_y(x) < 0$ iii) $S_y(x) = 0$

c) $S_x(y) = r(y)\left(1 - \dfrac{e^{-(y-x)^2}\, e^{-x^2}}{e^{-y^2}}\right) = 0$

$r(y)>0!$

$\Rightarrow \quad \dfrac{e^{-(y-x)^2}\, e^{-x^2}}{e^{-y^2}} = 1$

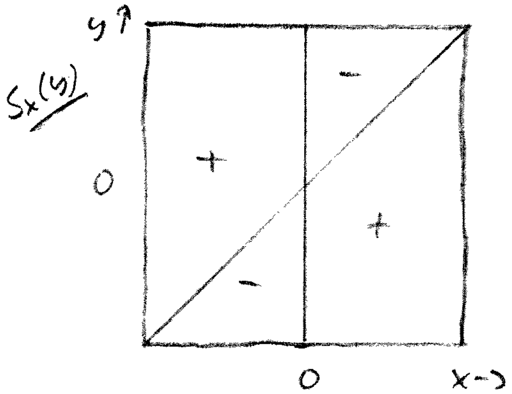
$\Rightarrow \quad e^{-(y-x)^2 - x^2 + y^2} = 1 \qquad \| \ln()$

$\Rightarrow \quad -(y-x)^2 - x^2 + y^2 = 0 \qquad\qquad (y-x)(y-x) = y^2 - 2xy + x^2$

$\Rightarrow \quad -y^2 + 2xy - x^2 - x^2 + y^2 = 0$

$\Rightarrow \quad 2xy - 2x^2 = 2x(y-x) = 0$

$\Rightarrow \quad \underline{x = 0} \quad \text{or} \quad \underline{y = x}$

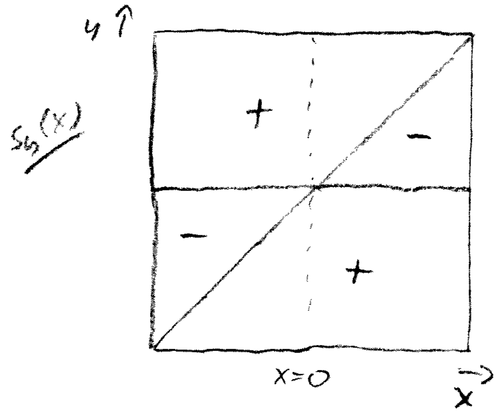


To calculate the sign of $S_x(y)$, take an example point, e.g.

$x=-1,\ y=0:\ S_{-1}(0) = r(y)\left(1 - \dfrac{e^{-1}e^{-1}}{e^{0}}\right)$

$= \underbrace{r(y)}_{>0}\underbrace{\left(1 - e^{-2}\right)}_{>0} > 0$

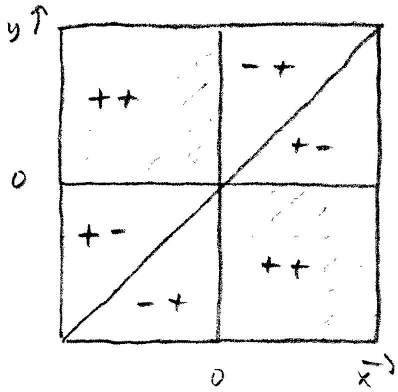
d) From the PIP we can immediately see that no strategy $x \neq 0$ is an ESS (there are strategies for which $S_x(y) > 0$, if $x \neq 0$). What about $x = 0$? Let's look at the sign-plot of $S_y(x)$:



For $x = 0$, $S_x(y) = 0$ for all y, but $S_y(x) > 0$ for all y. From 1b) we know that this implies stability.

$\therefore\ x^* = 0$ is the only ESS.

e) Let's combine the sign-plots of $s_x(y)$ and $s_y(x)$:



The points in the '++'-region fullfill the conditions $s_x(y)>0$ and $s_y(x)>0$, which implies coexistence.

f) The monomorphic population will converge to $x^*=0$ and remain there.

Exercise 2

$$K(x) = e^{-x^2}, \quad a(x,y) = e^{-\alpha|x-y|}, \quad \alpha > 0$$

a)
$$S_x(y) = \underbrace{r(y)}_{>0}\left(1 - \frac{e^{-\alpha|y-x|}\, e^{-x^2}}{e^{-y^2}}\right) = 0$$

$$\Rightarrow \quad e^{-\alpha|y-x|-x^2+y^2} = 1 \qquad || \ln()$$

$$\Rightarrow \; -\alpha|y-x|-x^2+y^2 = -\alpha|y-x| + (y-x)(y+x) = 0 \quad (*)$$

<u>$y > x$:</u>

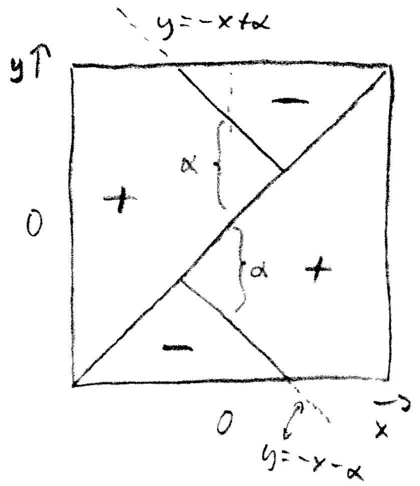
$$(*) \Rightarrow \; -\alpha(y-x) + (y-x)(y+x) = (y-x)\left[-\alpha+y+x\right] = 0$$

$$\Rightarrow \quad \underline{y = x} \quad \text{or} \quad \underline{y = -x+\alpha}$$

<u>$y < x$:</u>

$$(*) \Rightarrow \; \alpha(y-x) + (y-x)(y+x) = (y-x)\left[\alpha+y+x\right] = 0$$

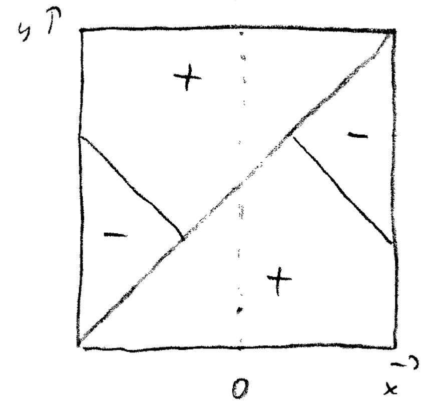
$$\Rightarrow \quad \underline{y = x} \quad \text{or} \quad \underline{y = -x-\alpha}$$



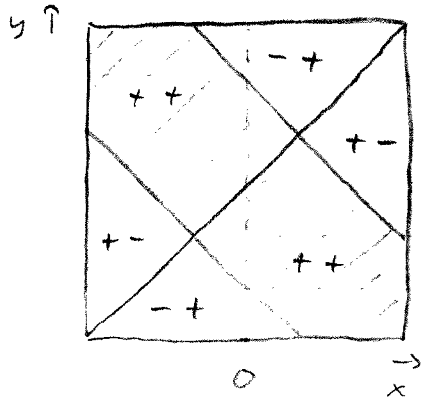
b) Changing 'α' moves the lines $y = -x \pm \alpha$ up or down. As 'α' tends to zero, the two lines come closer ($\alpha \to 0$, $y = -x \pm \alpha \to y = -x$).

c) From the PIP we can see that there are no evolutionarily stable strategies.

d) The sign-plot of $S_y(x)$ is:



Now we can combine the sign-plots of $S_x(y)$ and $S_y(x)$:



Strategies in the '++'-region can coexist.

e) Convergence to the '++'-region after which the population becomes dimorphic.

Exercise 3

a) i) clonal reproduction

ii) resident population is in a population attractor by the time the next mutant comes along

iii) initial mutant population density is very small compared to the resident population density

iv) small mutation steps

b) i) the resident-mutant dynamics is such that no intermediate offspring of the two is born and all the members of the population have either the resident or the mutant strategy, i.e., we only have two differential equations

$$\begin{cases} \dot{n} = \dots \\ \dot{m} = \dots \end{cases}$$

ii) when calculating invasion fitness $S_x(y)$, we take the limit $n \to K(x)$

iii) when calculating invasion fitness $S_x(y)$, we take the limit $m \to 0$

iv) the mutant strategy must be close to the resident strategy, i.e., in the PIPs we only need to consider the strip close to $x=y$ (if the population is monomorphic)