```
> s:=(x,y)->r(y)*(1-(a(y,x)*K(x))/K(y));  #invasion fitness
```

$$s := (x, y) \to \mathrm{r}(y)\left(1 - \frac{\mathrm{a}(y, x)\,\mathrm{K}(x)}{\mathrm{K}(y)}\right)$$

```
> K:=x->exp(-x^2);
```

$$K := x \to \mathbf{e}^{(-x^2)}$$

```
> a:=(x,y)->exp(-alpha*(x-y)^2);
```

$$a := (x, y) \to \mathbf{e}^{(-\alpha\,(x-y)^2)}$$

```
> r:=x->1;
```

$$r := x \to 1$$

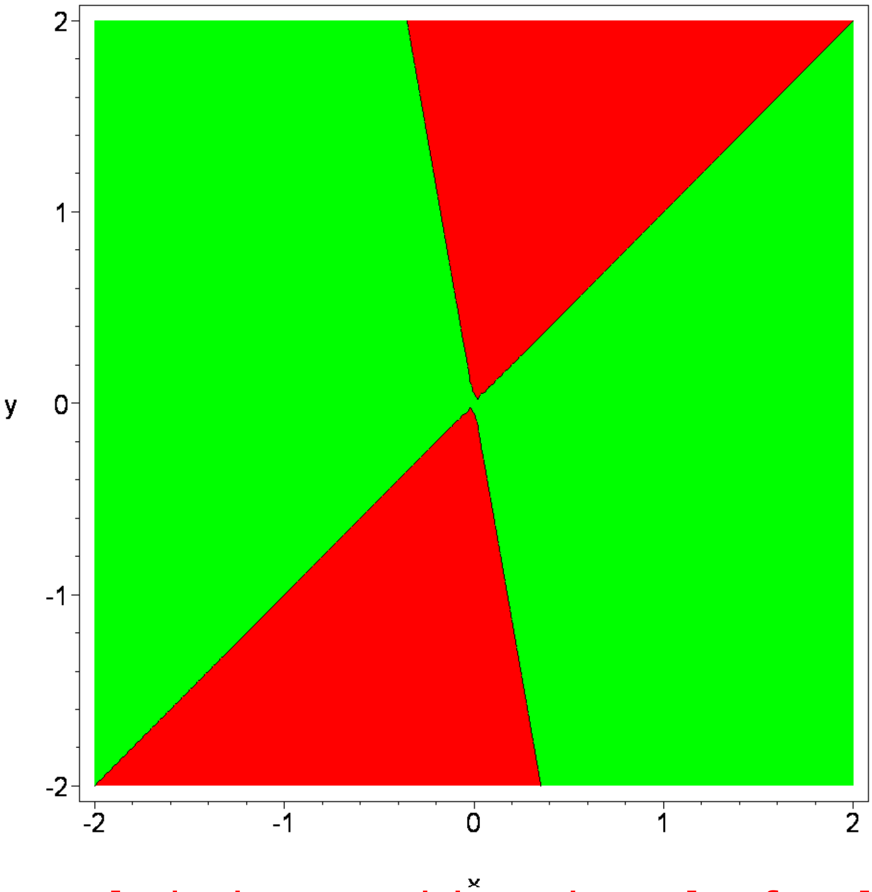
```
> s(x,x); #the fitness of the resident strategy in the res.pop.
  must be 0!! it's always good to check this
```

$$0$$

```
> alpha:=0.7;
```

$$\alpha := 0.7$$

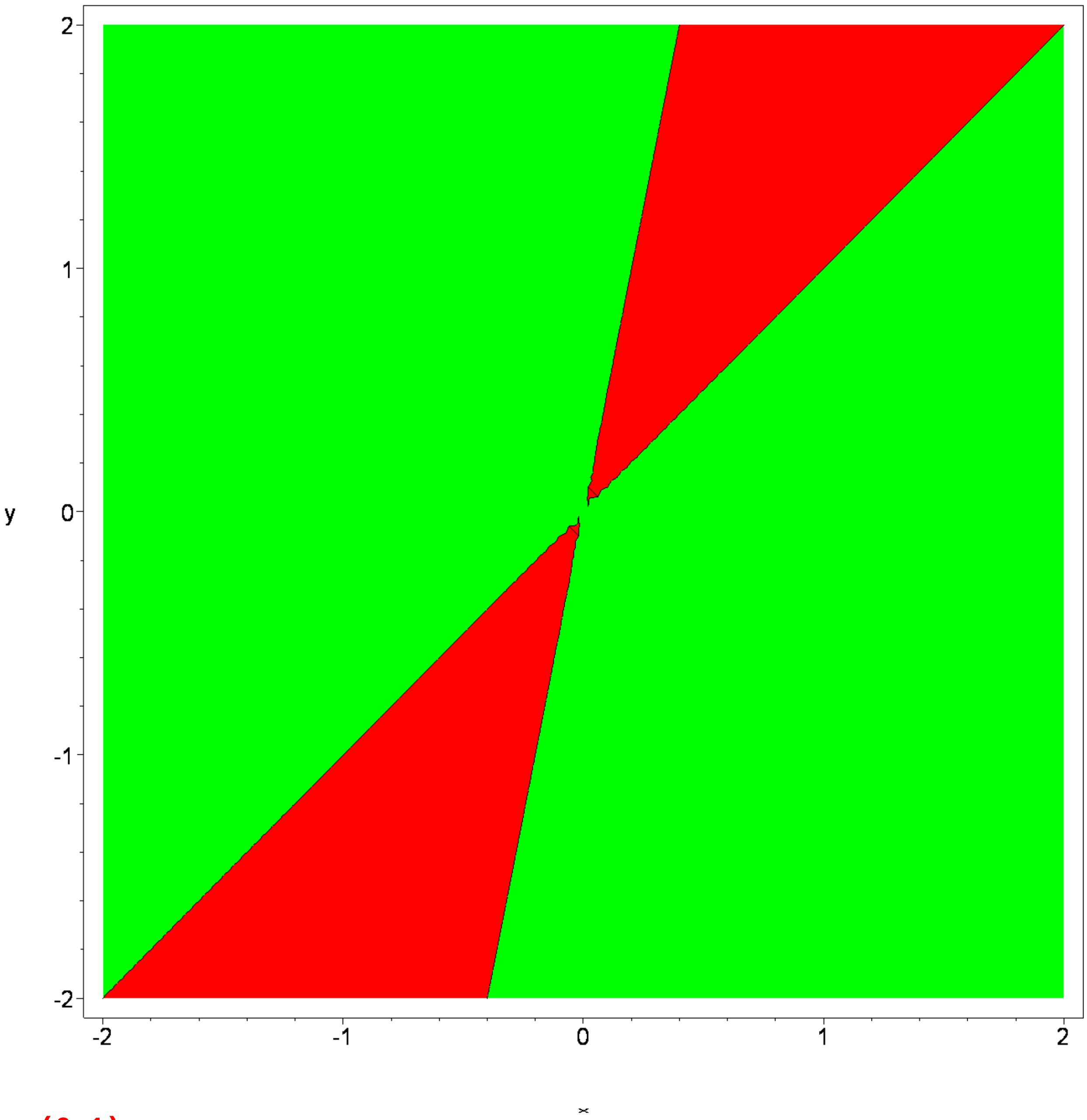
```
> with(plots):    #you need this to make, e.g., contourplots
> contourplot(s(x,y),x=-2..2,y=-2..2,contours=[0],filled=true,colo
  ring=[red,green],grid=[100,100],axes=boxed); #pairwise
  invadability plot (PIP)
```




```
> alpha:=1.5;  #let's do a PIP with another value for alpha
```

$$\alpha := 1.5$$

```
> contourplot(s(x,y),x=-2..2,y=-2..2,contours=[0],filled=true,colo
  ring=[red,green],grid=[100,100],axes=boxed);
```




```
> s(0,1);
```

$$1-\frac{0.2231301601}{e^{(-1)}}$$

```
> evalf(%);  #this evaluates the previous expression, we see that the colour green in the PIP indeed corresponds to s(x,y)>0
```

$$0.3934693405$$

```
> K:=x->exp(-(x-delta)^4)+exp(-(x+delta)^2);
```

$$K := x \to e^{(-(x-\delta)^4)} + e^{(-(x+\delta)^2)}$$

```
> a:=(x,y)->exp(-alpha*(x-y)^2-beta*(x-y));
```

$$a := (x, y) \to e^{(-\alpha\,(x-y)^2 - \beta\,(x-y))}$$

```
> alpha:=2;
```

$$\alpha := 2$$

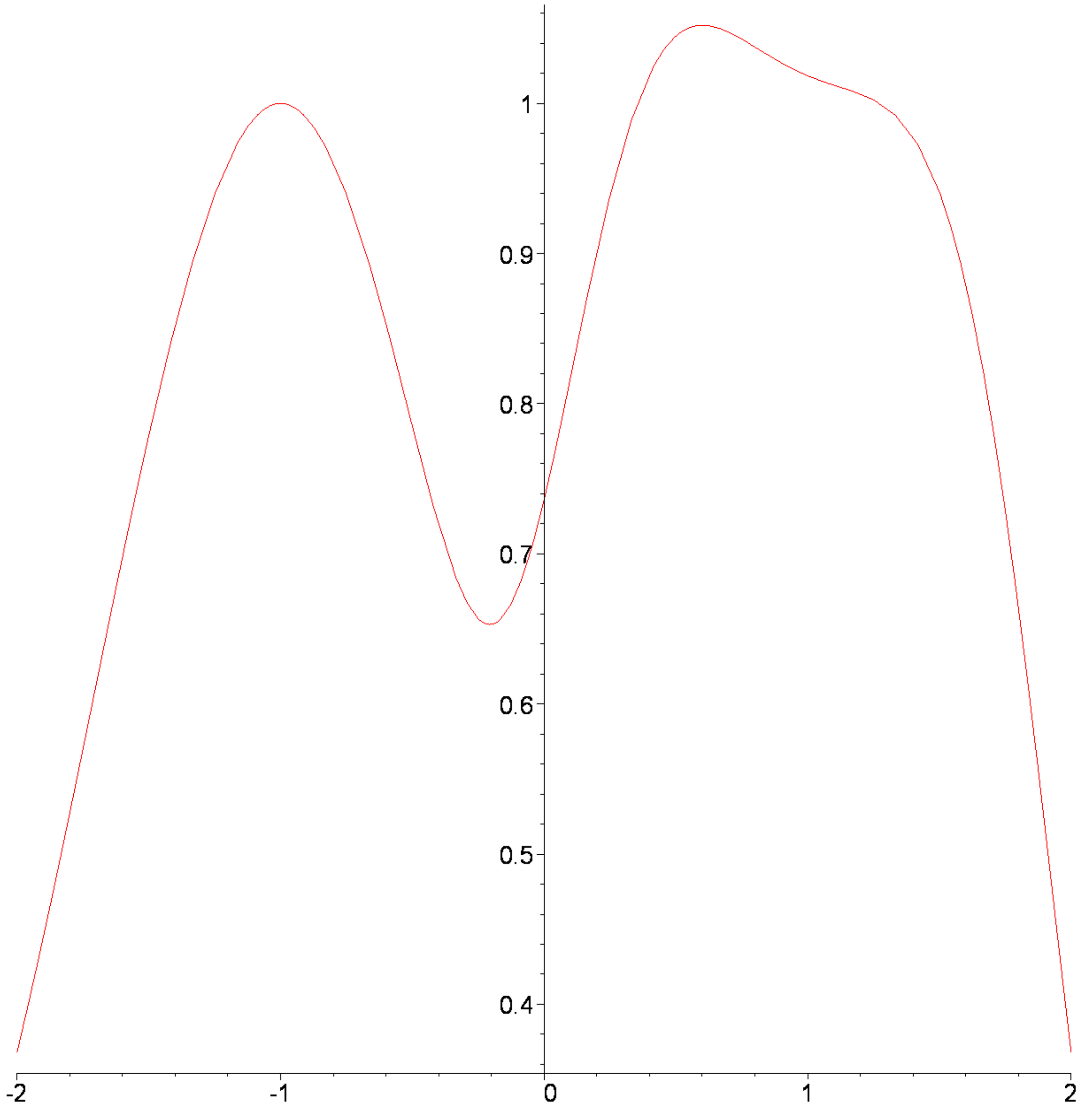
```
> beta:=-.4;
```

$\beta := -0.4$

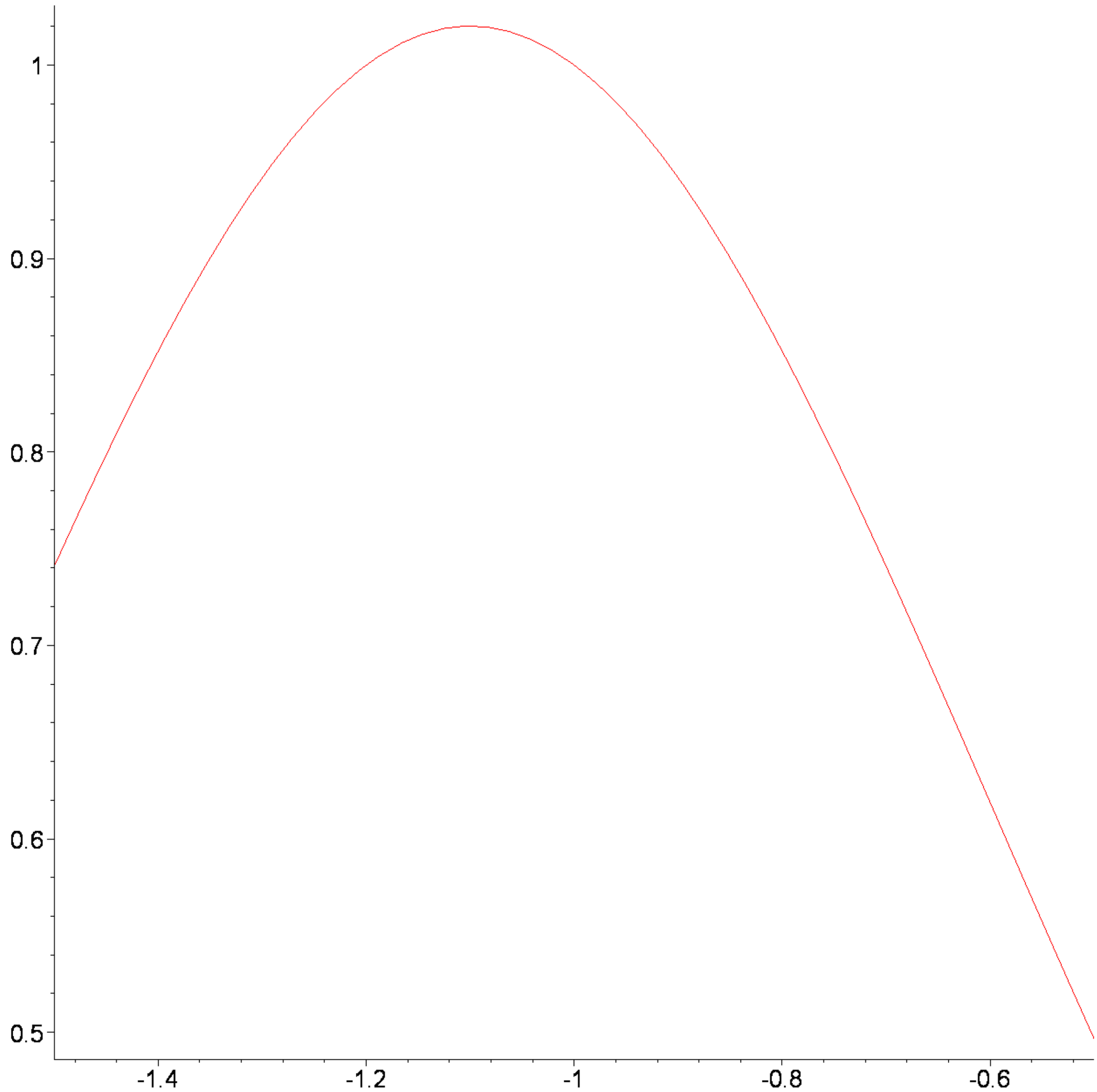
```
> ;delta:=1;
```

$\delta := 1$

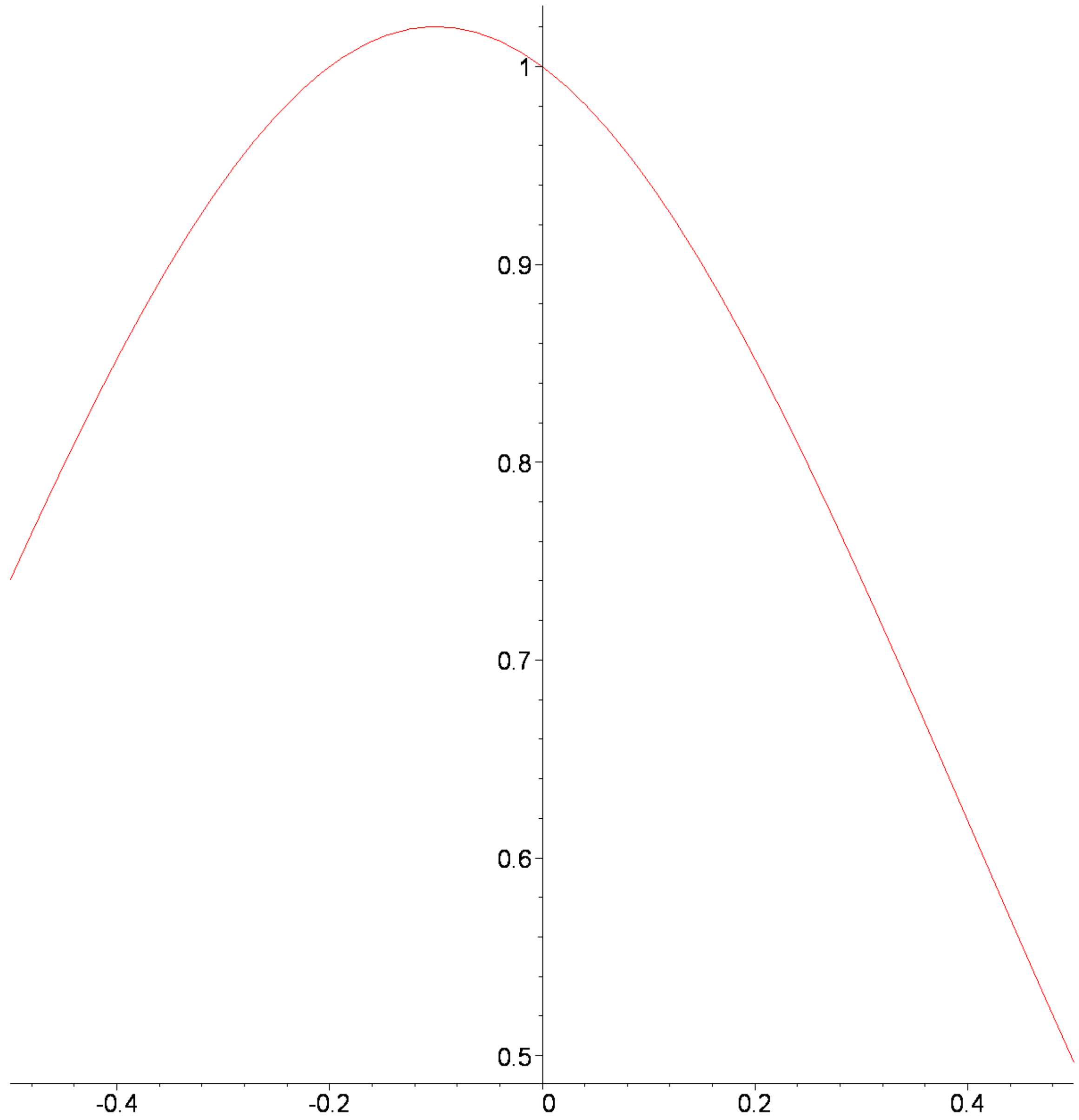
```
> plot(K(x),x=-2..2); #carrying capacity as a function of x, note
  that moving sufficiently far away from x=0, the carrying
  capacity goes towards zero, i.e., the population is less and
  less viable
```



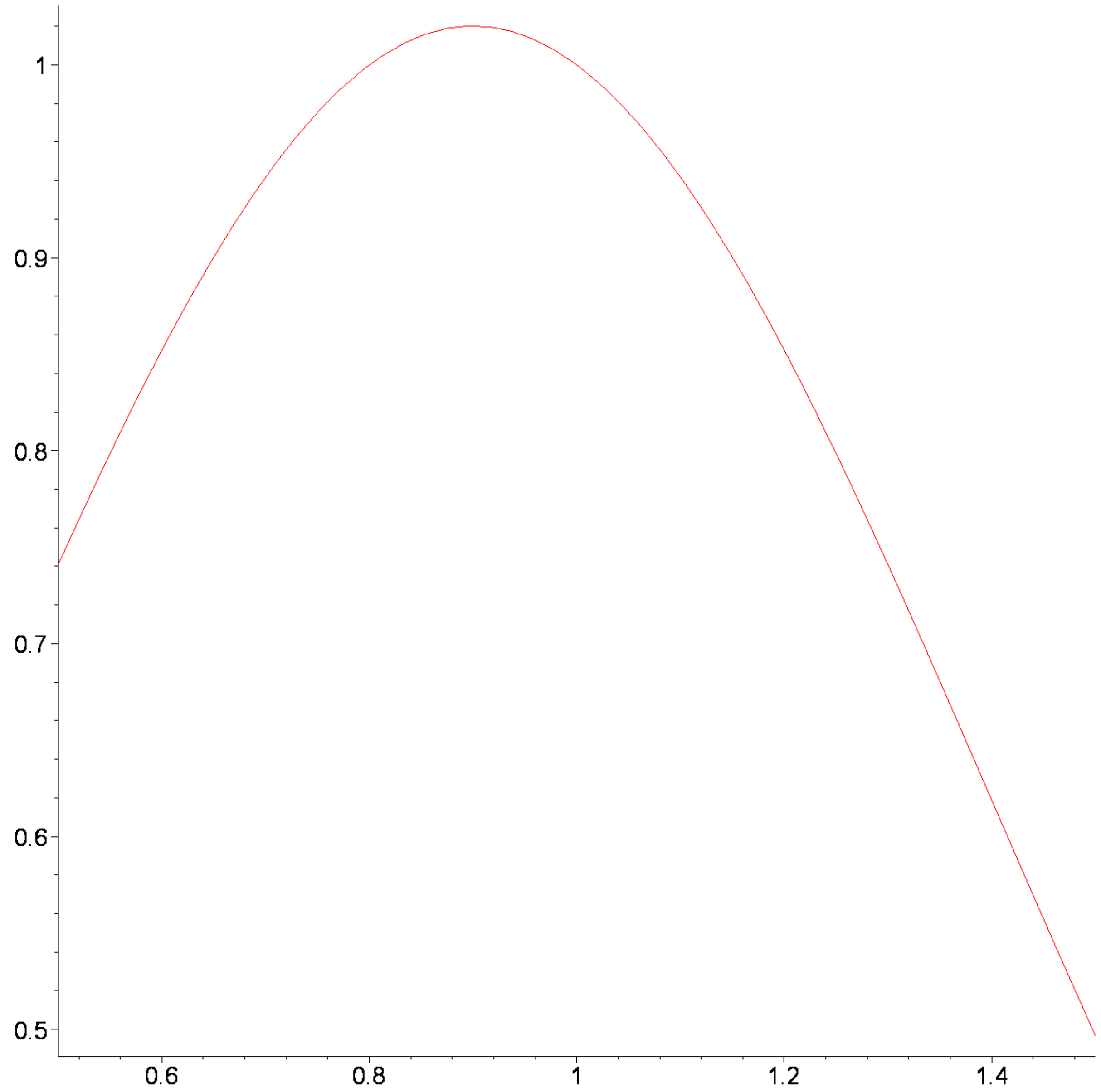

```
> plot(a(-1,y), y=-1.5..-0.5); #note how a(-1,y)>a(-1,-1) for some
  y<x close to x and a(-1,y)<a(-1,-1) for y>x -> competition is
  asymmetric!
```



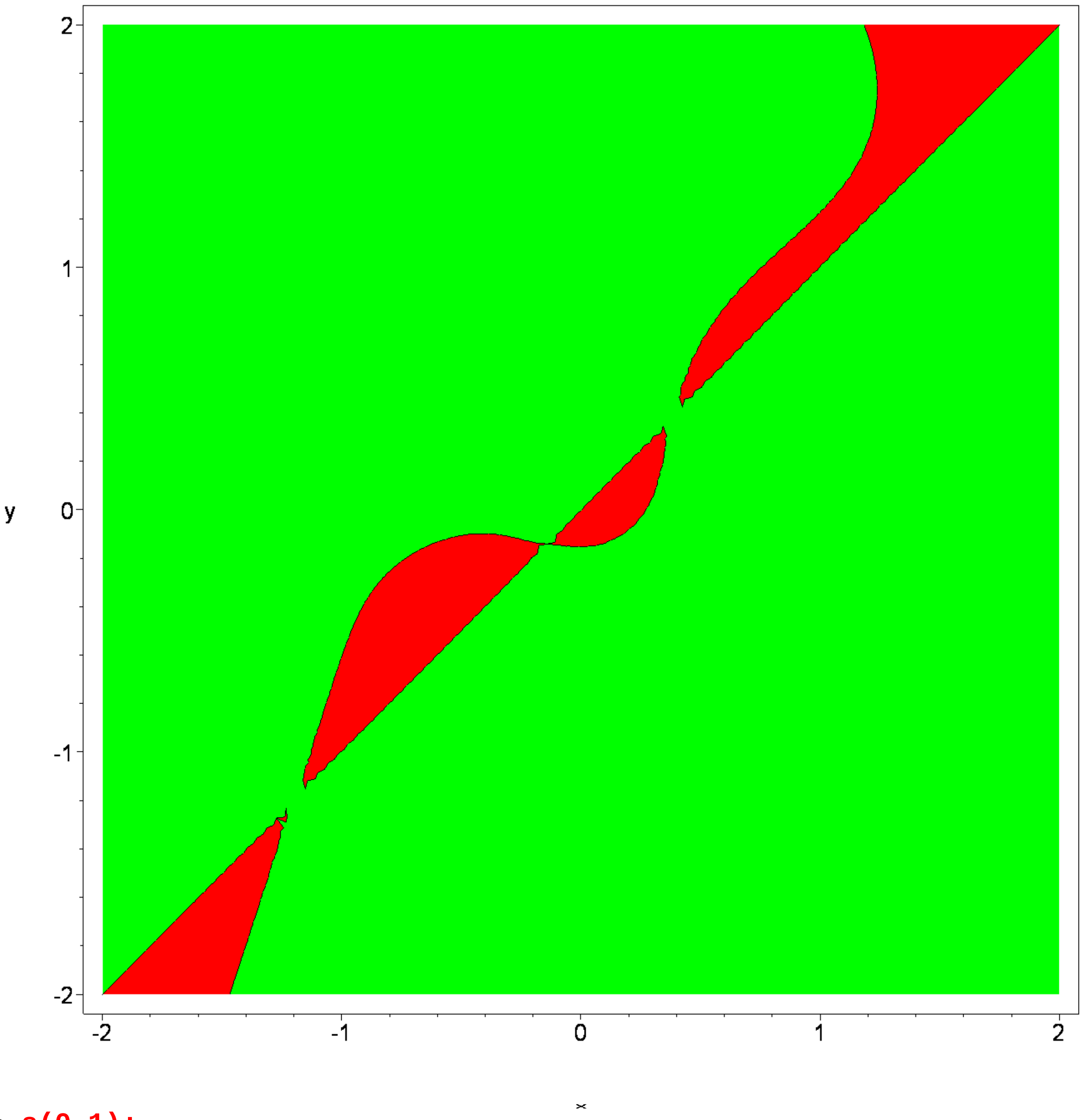

```
> plot(a(0,y),y=-.5..0.5);
```




```
> plot(a(1,y),y=.5..1.5);
```




```
> contourplot(s(x,y),x=-2..2,y=-2..2,contours=[0],filled=true,colo
  ring=[red,green],grid=[100,100],axes=boxed); #three
  singularities, two attractors and a repellor in between
```




```
> s(0,1);
```

$$1-\frac{0.4037930360\,\mathbf{e}^{(-1)}}{1+\mathbf{e}^{(-4)}}$$

```
> evalf(%);
```

$$0.8541246439$$

```
> restart; #clears the memory
> s:=(x,y,beta)->r(y)*(1-(a(y,x,beta)*K(x))/K(y)); #defines the invasion fitness as a function of x,y and beta!
```

$$s := (x, y, \beta) \rightarrow \mathrm{r}(y)\left(1-\frac{\mathrm{a}(y, x, \beta)\,\mathrm{K}(x)}{\mathrm{K}(y)}\right)$$

```
> a:=(x,y,beta)->exp(-alpha*(x-y)^2-beta*(x-y)); #now also the competition kernel is a function of x,y and beta
```

$$a := (x, y, \beta) \rightarrow \mathbf{e}^{(-\alpha\,(x-y)^2-\beta\,(x-y))}$$

```
> K:=x->exp(-(x-delta)^4)+exp(-(x+delta)^2); #K is still a function of x, delta is a constant
```

$$K := x \to \mathbf{e}^{(-(x-\delta)^4)} + \mathbf{e}^{(-(x+\delta)^2)}$$

```
> r:=x->1;
```

$$r := x \to 1$$

```
> s(x,y,beta); #just to see what the fitness function looks like
```

$$1 - \frac{\mathbf{e}^{(-\alpha\,(y-x)^2 - \beta\,(y-x))}\,(\mathbf{e}^{(-(x-\delta)^4)} + \mathbf{e}^{(-(x+\delta)^2)})}{\mathbf{e}^{(-(y-\delta)^4)} + \mathbf{e}^{(-(y+\delta)^2)}}$$

```
> s(x,x,beta);
```

$$0$$

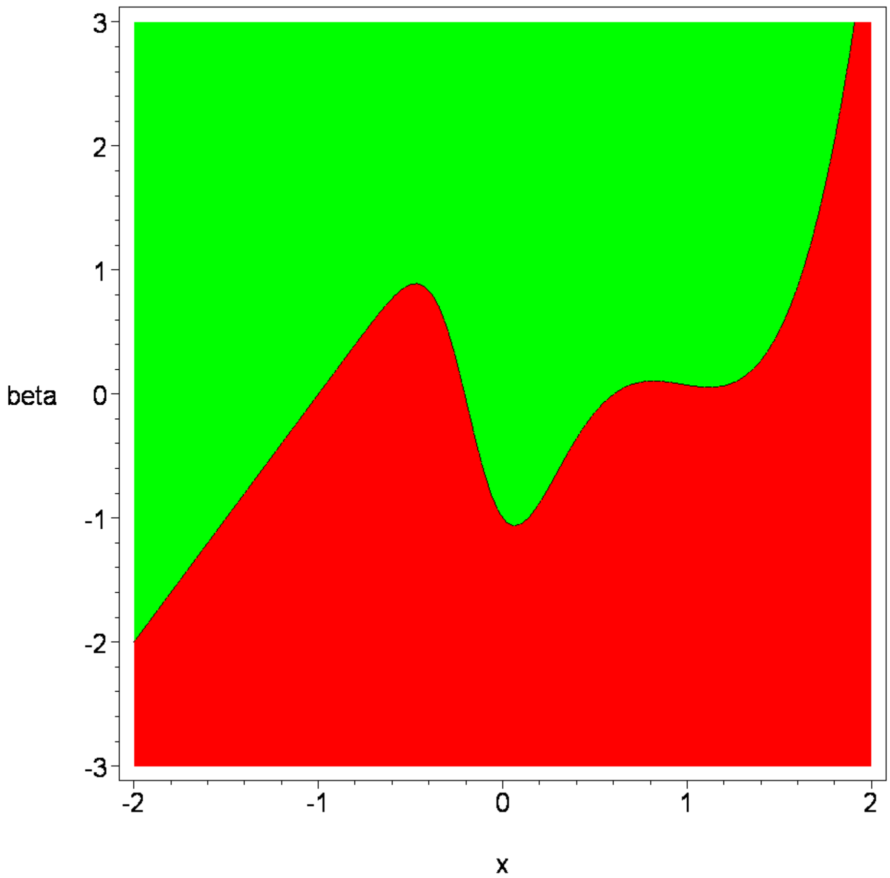
```
> eval(diff(s(x,y,beta),y),y=x); #differentiation of s(x,y,beta) with respect to y and then evaluating at y=x (this is the selection gradient!)
```

$$\beta + \frac{-4\,(x-\delta)^3\,\mathbf{e}^{(-(x-\delta)^4)} + (-2\,x - 2\,\delta)\,\mathbf{e}^{(-(x+\delta)^2)}}{\mathbf{e}^{(-(x-\delta)^4)} + \mathbf{e}^{(-(x+\delta)^2)}}$$

```
> Ds:=(x,beta)->beta+1/(exp(-(x-delta)^4)+exp(-(x+delta)^2))*(-4*(x-delta)^3*exp(-(x-delta)^4)+(-2*x-2*delta)*exp(-(x+delta)^2)); #defines the selection gradient as a function of x and beta
```

$$Ds := (x, \beta) \to \beta + \frac{-4\,(x-\delta)^3\,\mathbf{e}^{(-(x-\delta)^4)} + (-2\,x - 2\,\delta)\,\mathbf{e}^{(-(x+\delta)^2)}}{\mathbf{e}^{(-(x-\delta)^4)} + \mathbf{e}^{(-(x+\delta)^2)}}$$

```
> alpha:=2:delta:=1: #set the parameters
> with(plots):
> contourplot(Ds(x,beta),x=-2..2,beta=-3..3,contours=[0],filled=true,coloring=[red,green],grid=[100,100],axes=boxed); #Try reading this plot. Lets take, e.g., beta=2, which is a horizontal line in the plot. There is only one singularity for this parameter value at around x'=1.8. When x<x', then the selection gradient is positive and we have evolution to higher trait values (to the right). When x>x', the selection gradient is negative and the system evolves to the left. So, the singularity is attracting.
```




```
> eval(diff(s(x,y,beta),y$2),y=x); #the second derivative of
  s(x,y,beta) with respect to y, evaluated at y=x
```

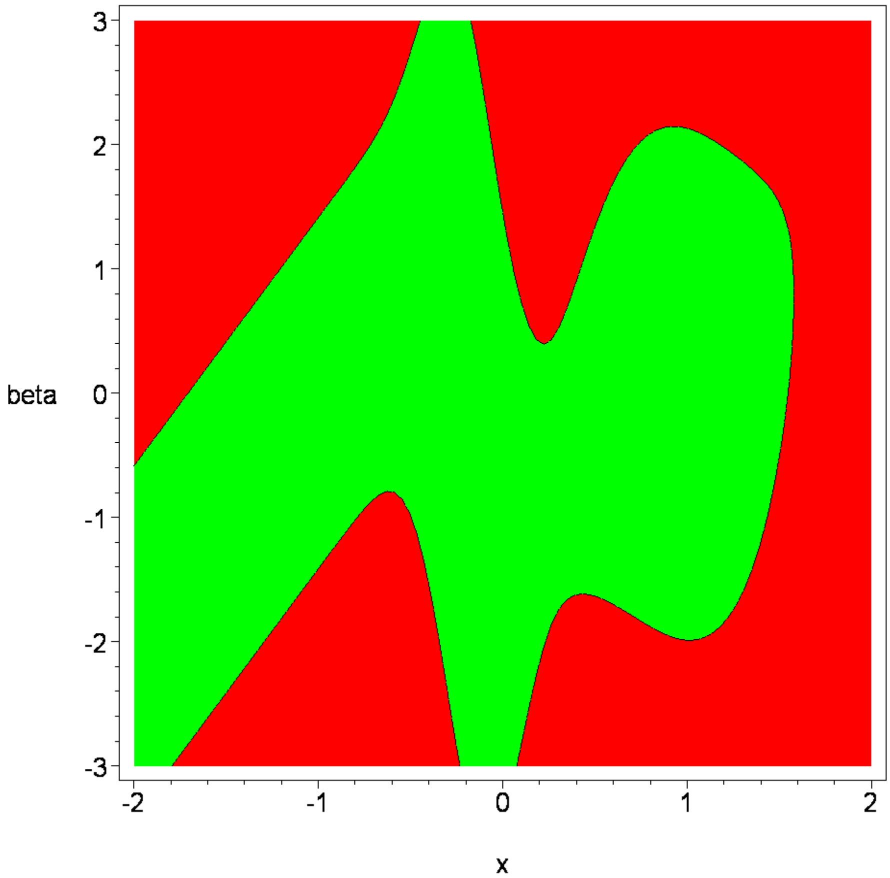
$$4-\beta^2-\frac{2\,\beta\,(-4\,(x-1)^3\,\mathbf{e}^{(-(x-1)^4)}+(-2\,x-2)\,\mathbf{e}^{(-(x+1)^2)})}{\mathbf{e}^{(-(x-1)^4)}+\mathbf{e}^{(-(x+1)^2)}}-\frac{2\,(-4\,(x-1)^3\,\mathbf{e}^{(-(x-1)^4)}+(-2\,x-2)\,\mathbf{e}^{(-(x+1)^2)})^2}{(\mathbf{e}^{(-(x-1)^4)}+\mathbf{e}^{(-(x+1)^2)})^2}+\frac{-12\,(x-1)^2\,\mathbf{e}^{(-(x-1)^4)}+16\,(x-1)^6\,\mathbf{e}^{(-(x-1)^4)}-2\,\mathbf{e}^{(-(x+1)^2)}+(-2\,x-2)^2\,\mathbf{e}^{(-(x+1)^2)}}{\mathbf{e}^{(-(x-1)^4)}+\mathbf{e}^{(-(x+1)^2)}}$$

```
> D2s:=(x,beta)->4-beta^2-2*beta/(exp(-(x-1)^4)+exp(-(x+1)^2))*(-4
  *(x-1)^3*exp(-(x-1)^4)+(-2*x-2)*exp(-(x+1)^2))-2/(exp(-(x-1)^4)+
  exp(-(x+1)^2))^2*(-4*(x-1)^3*exp(-(x-1)^4)+(-2*x-2)*exp(-(x+1)^2
  ))^2+1/(exp(-(x-1)^4)+exp(-(x+1)^2))*(-12*(x-1)^2*exp(-(x-1)^4)+
  16*(x-1)^6*exp(-(x-1)^4)-2*exp(-(x+1)^2)+(-2*x-2)^2*exp(-(x+1)^2
  ));
```

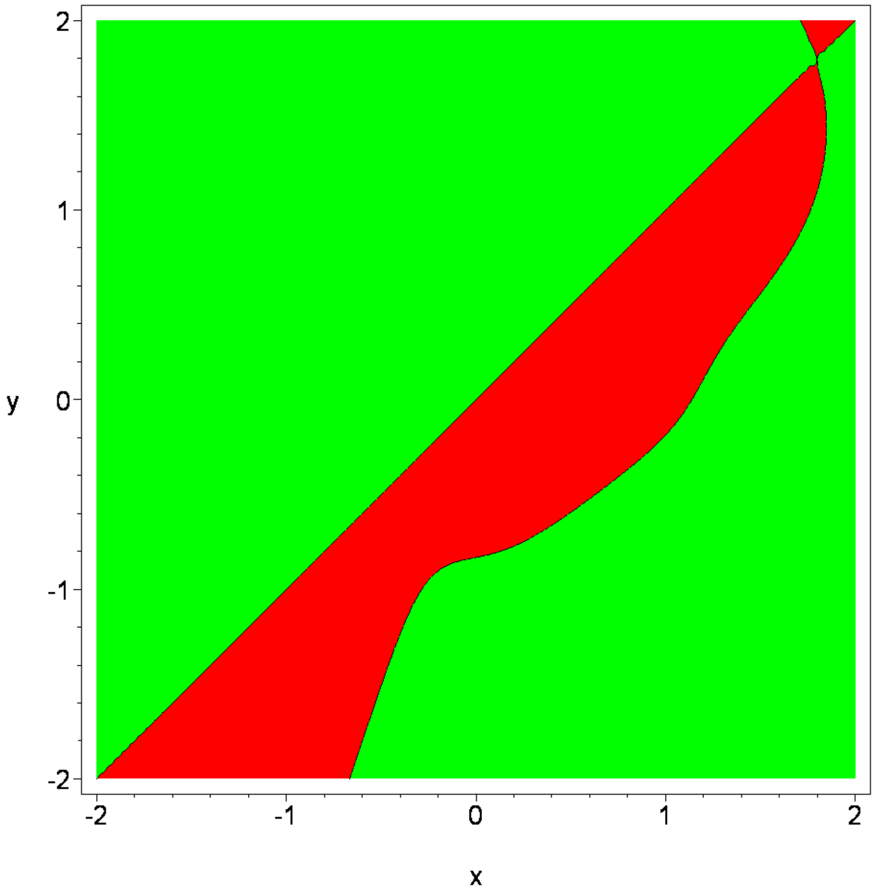
$$D2s:=(x,\beta)\rightarrow 4-\beta^2-\frac{2\,\beta\,(-4\,(x-1)^3\,\mathbf{e}^{(-(x-1)^4)}+(-2\,x-2)\,\mathbf{e}^{(-(x+1)^2)})}{\mathbf{e}^{(-(x-1)^4)}+\mathbf{e}^{(-(x+1)^2)}}$$

$$-\frac{2\left(-4(x-1)^3 \mathbf{e}^{(-(x-1)^4)}+(-2x-2)\,\mathbf{e}^{(-(x+1)^2)}\right)^2}{\left(\mathbf{e}^{(-(x-1)^4)}+\mathbf{e}^{(-(x+1)^2)}\right)^2}+\frac{-12(x-1)^2 \mathbf{e}^{(-(x-1)^4)}+16(x-1)^6 \mathbf{e}^{(-(x-1)^4)}-2\,\mathbf{e}^{(-(x+1)^2)}+(-2x-2)^2 \mathbf{e}^{(-(x+1)^2)}}{\mathbf{e}^{(-(x-1)^4)}+\mathbf{e}^{(-(x+1)^2)}}$$

```
> contourplot(D2s(x,beta),x=-2..2,beta=-3..3,contours=[0],filled=true,coloring=[red,green],grid=[100,100],axes=boxed);  #From the previous plot we saw that for beta=2, there is only one singularity and it is at about x=1.8. Here we can see that the point in question has a negative value of D2s; this means that it is dimorphically attracting (there is a minus region straight on top of and below the singularity in the PIP). If D2s was positive, the singularity would be dimorphically repelling.
```




```
> contourplot(s(x,y,2),x=-2..2,y=-2..2,contours=[0],filled=true,coloring=[red,green],grid=[100,100],axes=boxed);    #here is the PIP for beta=2 and voila, it looks just like we have predicted! There is only one singularity, which is monomorphically attracting. If we draw a vertical line through the singularity, we can see that nearby mutants have negative invasion fitness. The singularity is an ESS.
```



2
1
0
-1
-2
y
-2
-1
0
1
2
x

>

>