```
> s:=(x,y)->r(y)*(1-(a(y,x)*K(x))/K(y));  #invasion fitness
```

$$s := (x, y) \;/\; r(y) \left(1 - \frac{a(y, x)\, K(x)}{K(y)} \right) \tag{1}$$

```
> K:=x->exp(-(x-delta)^4)+exp(-(x+delta)^2);
```

$$K := x \;/\; e^{-(x - d)^4} + e^{-(x + d)^2} \tag{2}$$

```
> a:=(x,y)->exp(-alpha*(x-y)^2-beta*(x-y));
```

$$a := (x, y) \;/\; e^{-a\,(x - y)^2 - b\,(x - y)} \tag{3}$$

```
> r:=x->1;
```

$$r := x \;/\; 1 \tag{4}$$

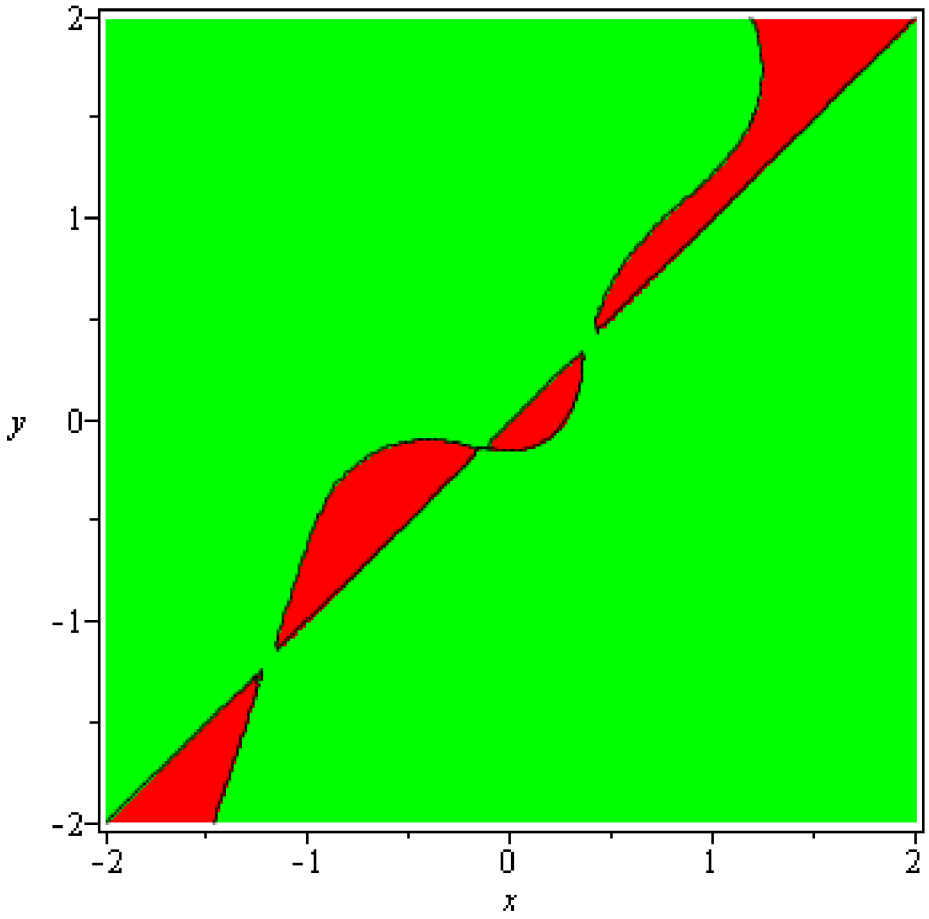
```
> alpha:=2;beta:=-0.4;delta:=1;
```

$$a := 2$$

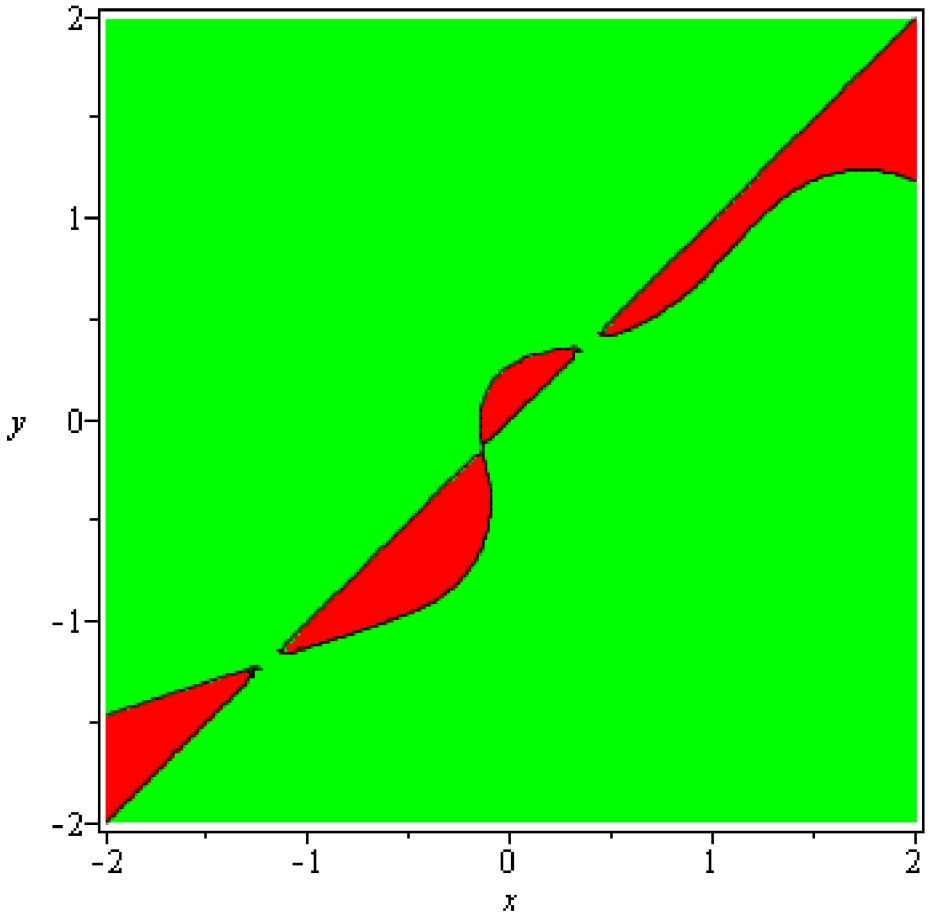
$$b := -0.4$$

$$d := 1 \tag{5}$$

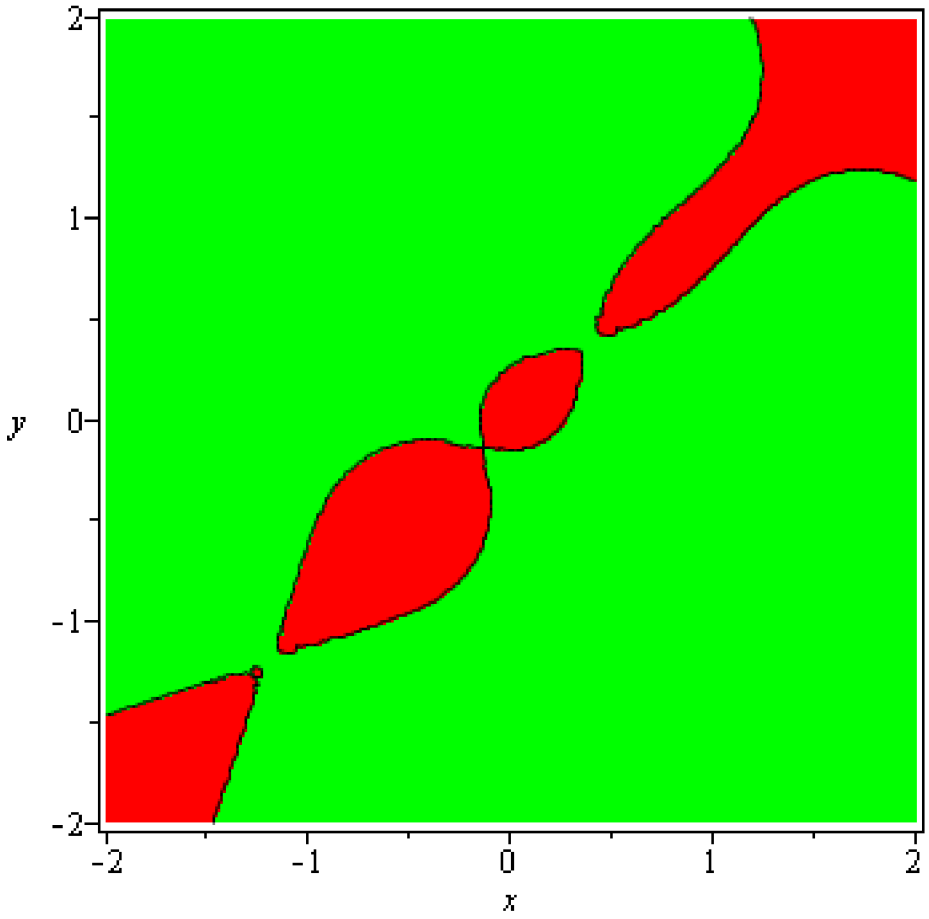
```
> with(plots):
> contourplot(s(x,y),x=-2..2,y=-2..2,contours=[0],filled=true,
  coloring=[red,green],grid=[100,100],axes=boxed); #pairwise
  invadability plot (PIP)
```




```
> contourplot(s(y,x),x=-2..2,y=-2..2,contours=[0],filled=true,
  coloring=[red,green],grid=[100,100],axes=boxed);
```




```
O contourplot(s(x,y)*s(y,x),x=-2..2,y=-2..2,contours=[0],filled=
  true,coloring=[red,green],grid=[100,100],axes=boxed); #MIP
```




```
restart;
k:=2;
```

$$k := 2 \tag{6}$$

```
for i from 1 to k do
1-(( sum(a(x[i],x[j])*n[j], j=1..k) )/K(x[i]))=0
end do;
```

$$1 - \frac{a(x_1, x_1)\, n_1 + a(x_1, x_2)\, n_2}{K(x_1)} = 0$$

$$1 - \frac{a(x_2, x_1)\, n_1 + a(x_2, x_2)\, n_2}{K(x_2)} = 0 \tag{7}$$

```
solve([1-(a(x[1],x[1])*n[1]+a(x[1],x[2])*n[2])/K(x[1]) = 0, 1-(a
(x[2],x[1])*n[1]+a(x[2],x[2])*n[2])/K(x[2]) = 0],[n[1],n[2]]);
```

$$\left[\left[n_1 = \frac{-a(x_2, x_2)\, K(x_1) + K(x_2)\, a(x_1, x_2)}{-a(x_1, x_1)\, a(x_2, x_2) + a(x_1, x_2)\, a(x_2, x_1)}, n_2 = \right.\right. \tag{8}$$

$$- \frac{K(x_1)\, a(x_2, x_1) + a(x_1, x_1)\, K(x_2)}{a(x_1, x_1)\, a(x_2, x_2) + a(x_1, x_2)\, a(x_2, x_1)} \Big] \Big]$$

```
> subs(x[1]=x,x[2]=y,%);
```

$$\left[\left[n_1 = \frac{-a(y,y)\,K(x) + K(y)\,a(x,y)}{-a(x,x)\,a(y,y) + a(x,y)\,a(y,x)},\ n_2 = -\frac{-K(x)\,a(y,x) + a(x,x)\,K(y)}{-a(x,x)\,a(y,y) + a(x,y)\,a(y,x)}\right]\right] \quad (9)$$

```
> n[1]:=(x,y)->(-a(y,y)*K(x)+K(y)*a(x,y))/(-a(x,x)*a(y,y)+a(x,y)*a
  (y,x));
```

$$n_1 := (x, y) \mapsto \frac{-a(y,y)\,K(x) + K(y)\,a(x,y)}{-a(x,x)\,a(y,y) + a(x,y)\,a(y,x)} \quad (10)$$

```
> n[2]:=(x,y)->-1/(-a(x,x)*a(y,y)+a(x,y)*a(y,x))*(-K(x)*a(y,x)+a
  (x,x)*K(y));
```

$$n_2 := (x, y) \mapsto -\frac{-K(x)\,a(y,x) + a(x,x)\,K(y)}{-a(x,x)\,a(y,y) + a(x,y)\,a(y,x)} \quad (11)$$

```
> K:=x->exp(-(x-delta)^4)+exp(-(x+delta)^2);
```

$$K := x \mapsto e^{-(x-d)^4} + e^{-(x+d)^2} \quad (12)$$

```
> a:=(x,y)->exp(-alpha*(x-y)^2-beta*(x-y));
```

$$a := (x, y) \mapsto e^{-a\,(x-y)^2 - b\,(x-y)} \quad (13)$$

```
> r:=x->1;
```

$$r := x \mapsto 1 \quad (14)$$

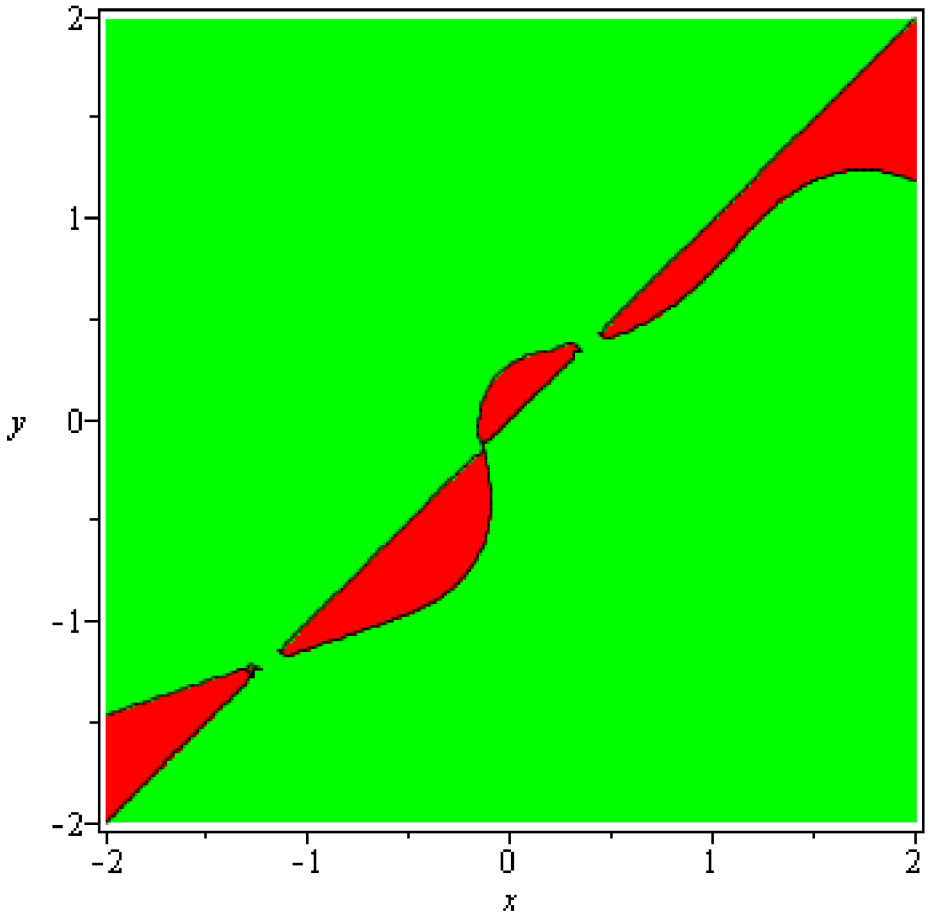
```
> alpha:=2;beta:=-0.4;delta:=1;
```

$$a := 2$$

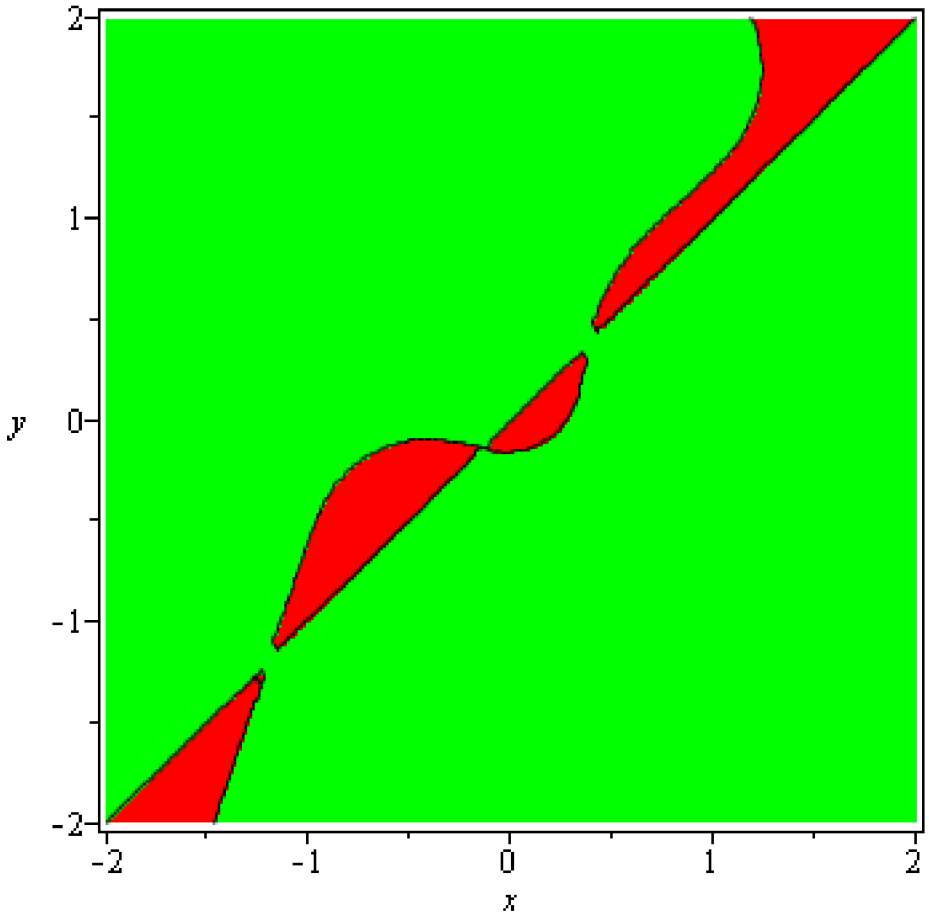
$$b := -0.4$$

$$d := 1 \quad (15)$$

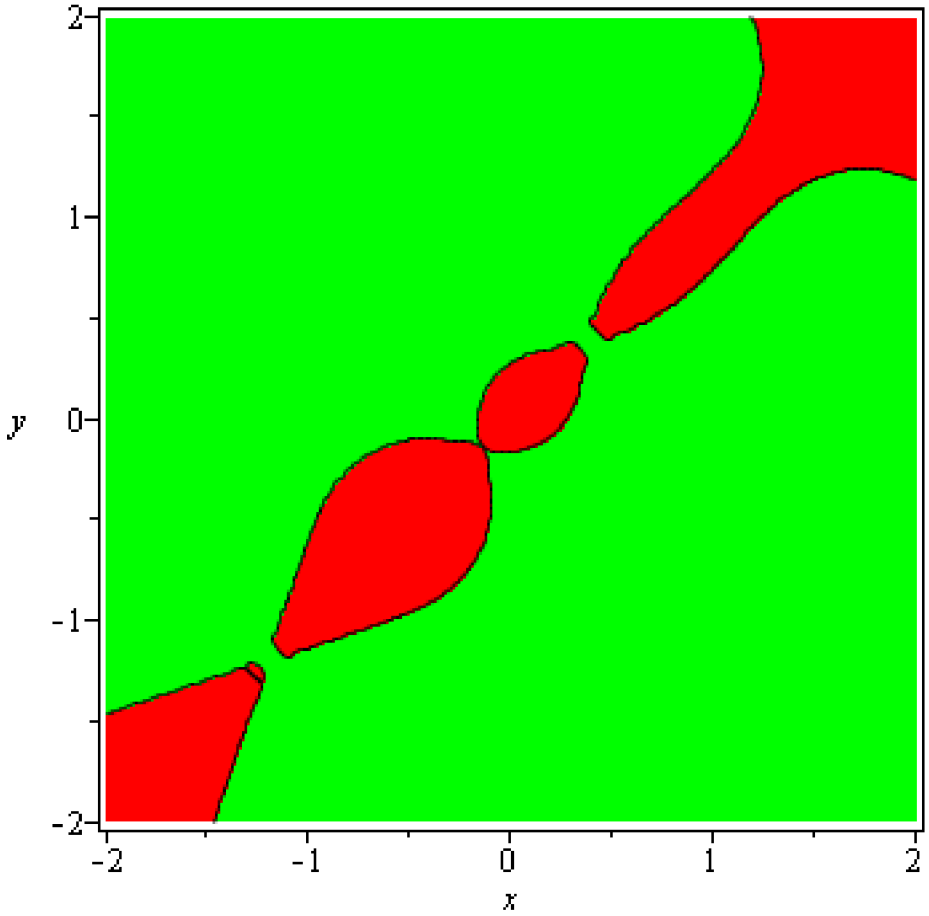
```
> with(plots):
> contourplot(piecewise(abs(x-y)<0.01,0,n[1](x,y)),x=-2..2,y=-2.
  .2,contours=[0],filled=true,coloring=[red,green],grid=[100,100],
  axes=boxed);
```




```
> contourplot(piecewise(abs(x-y)<0.01,0,n[2](x,y)),x=-2..2,y=-2.
  .2,contours=[0],filled=true,coloring=[red,green],grid=[100,100],
  axes=boxed);
```



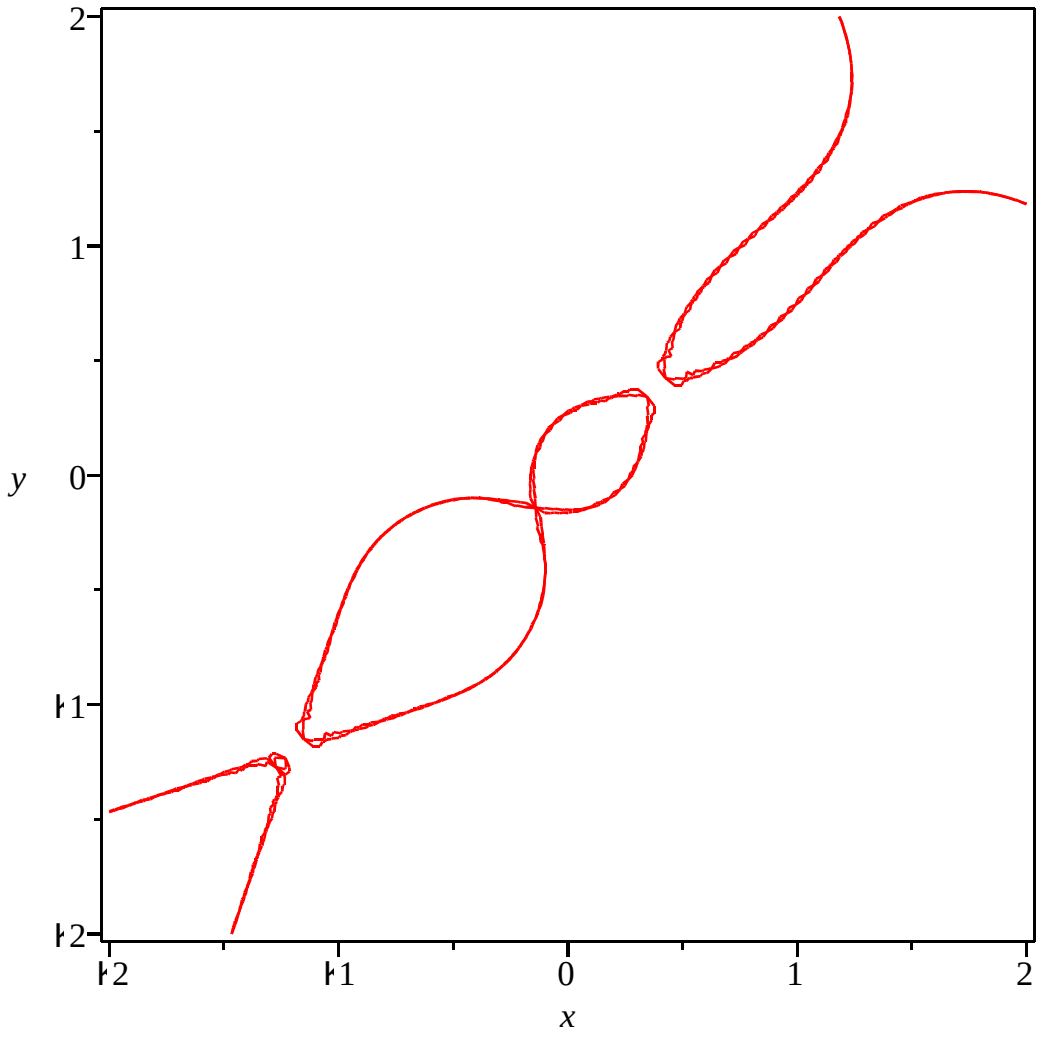

```
> contourplot(piecewise(abs(x-y)<0.01,0,n[1](x,y)*n[2](x,y)),x=-2.
  .2,y=-2..2,contours=[0],filled=true,coloring=[red,green],grid=
  [100,100],axes=boxed);
```




```
> s := (x,y) -> r(y)*(1-(a(y,x)*K(x))/K(y));
```

$$s := (x, y) \rightarrow r(y)\left(1 - \frac{K(x)\, a(y, x)}{K(y)}\right) \tag{16}$$

```
> M := contourplot(s(x,y)*s(y,x), x=-2..2, y=-2..2, contours=[0], grid=[100,100], axes=boxed): #MIP
> N := contourplot(piecewise(abs(x-y)<0.01, 0, n[1](x,y)*n[2](x,y)), x=-2..2, y=-2..2, contours=[0], grid=[100,100], axes=boxed):
> display({M,N});
```




```
> restart;
> s:=(x,y,z)->r(z)*( 1-((a(z,x)*n[1](x,y)+a(z,y)*n[2](x,y))/K(z))
  ); #invasion fitness of a rare mutant z in a resident population
  of x and y
```

$$s := (x, y, z) \mapsto r(z) \left(1 - \frac{a(z, x)\, n_1(x, y) + a(z, y)\, n_2(x, y)}{K(z)} \right) \tag{17}$$

```
> K:=x->exp(-(x-delta)^4)+exp(-(x+delta)^2);
```

$$K := x \mapsto e^{-(x - d)^4} + e^{-(x + d)^2} \tag{18}$$

```
> a:=(x,y)->exp(-alpha*(x-y)^2-beta*(x-y));
```

$$a := (x, y) \mapsto e^{-a\,(x - y)^2 - b\,(x - y)} \tag{19}$$

```
> r:=x->1;
```

$$r := x \mapsto 1 \tag{20}$$

```
> n[1]:=(x,y)->(-a(y,y)*K(x)+K(y)*a(x,y))/(-a(x,x)*a(y,y)+a(x,y)*a
  (y,x));
```

(21)

$$n_1 := (x, y) \;/\; \frac{-a(y,y)\,K(x) + K(y)\,a(x,y)}{-a(x,x)\,a(y,y) + a(x,y)\,a(y,x)} \tag{21}$$

```
n[2]:=(x,y)->-1/(-a(x,x)*a(y,y)+a(x,y)*a(y,x))*(-K(x)*a(y,x)+a(x,x)*K(y));
```

$$n_2 := (x, y) \;/\; -\frac{-K(x)\,a(y,x) + a(x,x)\,K(y)}{-a(x,x)\,a(y,y) + a(x,y)\,a(y,x)} \tag{22}$$

```
diff(s(x,y,z),z);
```

$$-\frac{1}{e^{-(z-d)^4} + e^{-(z+d)^2}}\left(\frac{1}{-1 + e^{-a(x-y)^2 - b(x-y)}\, e^{-a(y-x)^2 - b(y-x)}}\left(\left(-2\,a\,(z-x) - b\right) e^{-a(z-x)^2 - b(z-x)}\left(-e^{-(x-d)^4} - e^{-(x+d)^2}\right) e^{-a(x-y)^2 - b(x-y)} + \left(-e^{-(y-d)^4} - e^{-(y+d)^2}\right) e^{-a(y-x)^2 - b(y-x)}\right)\right.$$

$$-\frac{1}{-1 + e^{-a(x-y)^2 - b(x-y)}\, e^{-a(y-x)^2 - b(y-x)}}\left(\left(-2\,a\,(z-y) - b\right) e^{-a(z-y)^2 - b(z-y)}\left(-\left(e^{-(x-d)^4} + e^{-(x+d)^2}\right) e^{-a(y-x)^2 - b(y-x)} - e^{-(y-d)^4} - e^{-(y+d)^2}\right)\right)$$

$$+\frac{1}{\left(e^{-(z-d)^4} + e^{-(z+d)^2}\right)^2}\left(\left(1 \Big/ \left(-1 + e^{-a(x-y)^2 - b(x-y)}\, e^{-a(y-x)^2 - b(y-x)}\right)\left(e^{-a(z-x)^2 - b(z-x)}\left(-e^{-(x-d)^4} - e^{-(x+d)^2}\right) e^{-a(x-y)^2 - b(x-y)} + \left(e^{-(y-d)^4} + e^{-(y+d)^2}\right) e^{-a(y-x)^2 - b(y-x)}\right)\right.\right.$$

$$-\frac{1}{-1 + e^{-a(x-y)^2 - b(x-y)}\, e^{-a(y-x)^2 - b(y-x)}}\left(e^{-a(z-y)^2 - b(z-y)}\left(-\left(e^{-(x-d)^4} + e^{-(x+d)^2}\right) e^{-a(y-x)^2 - b(y-x)} - e^{-(y-d)^4} - e^{-(y+d)^2}\right)\right)\left(-4\,(z-d)^3 e^{-(z-d)^4} + \left(-2\,z - 2\,d\right) e^{-(z+d)^2}\right)\left.\right) \tag{23}$$

```
subs(z=x,%);
```

$$-\frac{1}{e^{-(x-d)^4} + e^{-(x+d)^2}}\left(\right. \tag{24}$$

$$-\frac{b\,e^{0}\left(-e^{-(x-d)^4}-e^{-(x+d)^2}+\left(e^{-(y-d)^4}+e^{-(y+d)^2}\right)e^{-a(x-y)^2-b(x-y)}\right)}{-1+e^{-a(x-y)^2-b(x-y)}\,e^{-a(y-x)^2-b(y-x)}}$$

$$-\frac{1}{-1+e^{-a(x-y)^2-b(x-y)}\,e^{-a(y-x)^2-b(y-x)}}\left(\left(-2a(x-y)-b\right)e^{-a(x-y)^2-b(x-y)}\left(-\left(e^{-(x-d)^4}+e^{-(x+d)^2}\right)e^{-a(y-x)^2-b(y-x)}+e^{-(y-d)^4}+e^{-(y+d)^2}\right)\right)$$

$$+\frac{1}{\left(e^{-(x-d)^4}+e^{-(x+d)^2}\right)^2}\left(\left(\frac{1}{-1+e^{-a(x-y)^2-b(x-y)}\,e^{-a(y-x)^2-b(y-x)}}\left(e^{0}\left(-e^{-(x-d)^4}-e^{-(x+d)^2}+\left(e^{-(y-d)^4}+e^{-(y+d)^2}\right)e^{-a(x-y)^2-b(x-y)}\right)\right)\right.\right.$$

$$-\frac{1}{-1+e^{-a(x-y)^2-b(x-y)}\,e^{-a(y-x)^2-b(y-x)}}\left(e^{-a(x-y)^2-b(x-y)}\left(-\left(e^{-(x-d)^4}+e^{-(x+d)^2}\right)e^{-a(y-x)^2-b(y-x)}+e^{-(y-d)^4}+e^{-(y+d)^2}\right)\right)\Bigg)\left(-4(x-d)^3e^{-(x-d)^4}+\left(-2x-2d\right)e^{-(x+d)^2}\right)\Bigg)$$

```
Ds[x]:=(x,y)->-(-beta*exp(0)*(-exp(-(x-delta)^4)-exp(-(x+delta)
^2)+(exp(-(y-delta)^4)+exp(-(y+delta)^2))*exp(-alpha*(x-y)^2-
beta*(x-y)))/(-1+exp(-alpha*(x-y)^2-beta*(x-y))*exp(-alpha*(y-x)
^2-beta*(y-x)))-(-2*alpha*(x-y)-beta)*exp(-alpha*(x-y)^2-beta*
(x-y))/(-1+exp(-alpha*(x-y)^2-beta*(x-y))*exp(-alpha*(y-x)^2-
beta*(y-x)))*(-(exp(-(x-delta)^4)+exp(-(x+delta)^2))*exp(-alpha*
(y-x)^2-beta*(y-x))+exp(-(y-delta)^4)+exp(-(y+delta)^2)))/(exp(-
(x-delta)^4)+exp(-(x+delta)^2))+(exp(0)*(-exp(-(x-delta)^4)-exp
(-(x+delta)^2)+(exp(-(y-delta)^4)+exp(-(y+delta)^2))*exp(-alpha*
(x-y)^2-beta*(x-y)))/(-1+exp(-alpha*(x-y)^2-beta*(x-y))*exp(-
alpha*(y-x)^2-beta*(y-x)))-exp(-alpha*(x-y)^2-beta*(x-y))/(-1+
exp(-alpha*(x-y)^2-beta*(x-y))*exp(-alpha*(y-x)^2-beta*(y-x)))*
(-(exp(-(x-delta)^4)+exp(-(x+delta)^2))*exp(-alpha*(y-x)^2-beta*
(y-x))+exp(-(y-delta)^4)+exp(-(y+delta)^2)))/(exp(-(x-delta)^4)+
exp(-(x+delta)^2))^2*(-4*(x-delta)^3*exp(-(x-delta)^4)+(-2*x-2*
delta)*exp(-(x+delta)^2));
```

$$Ds_x := (x,y) \;/\; -\frac{1}{e^{-(x-d)^4}+e^{-(x+d)^2}}\left(\tag{25}$$

$$-\frac{b e^{0} \left(-e^{-(x-d)^4} - e^{-(x+d)^2} + \left(e^{-(y-d)^4} + e^{-(y+d)^2}\right) e^{-a (x-y)^2 - b (x-y)}\right)}{-1 + e^{-a (x-y)^2 - b (x-y)}\, e^{-a (y-x)^2 - b (y-x)}}$$

$$-\frac{1}{-1 + e^{-a (x-y)^2 - b (x-y)}\, e^{-a (y-x)^2 - b (y-x)}} \left(\left(-2 a (x-y) - b\right) e^{-a (x-y)^2 - b (x-y)} \left(-\left(e^{-(x-d)^4} + e^{-(x+d)^2}\right) e^{-a (y-x)^2 - b (y-x)} + e^{-(y-d)^4} + e^{-(y+d)^2}\right)\right)$$

$$+\frac{1}{\left(e^{-(x-d)^4} + e^{-(x+d)^2}\right)^2} \left(\left(\frac{1}{-1 + e^{-a (x-y)^2 - b (x-y)}\, e^{-a (y-x)^2 - b (y-x)}} \left(e^{0} \left(-e^{-(x-d)^4} - e^{-(x+d)^2} + \left(e^{-(y-d)^4} + e^{-(y+d)^2}\right) e^{-a (x-y)^2 - b (x-y)}\right)\right)\right.\right.$$

$$-\frac{1}{-1 + e^{-a (x-y)^2 - b (x-y)}\, e^{-a (y-x)^2 - b (y-x)}} \left(e^{-a (x-y)^2 - b (x-y)} \left(-\left(e^{-(x-d)^4} + e^{-(x+d)^2}\right) e^{-a (y-x)^2 - b (y-x)} + e^{-(y-d)^4} + e^{-(y+d)^2}\right)\right)\left.\right) \left(-4 (x-d)^3 e^{-(x-d)^4} + \left(-2 x - 2 d\right) e^{-(x+d)^2}\right)\Bigg)$$

```
O subs(z=y,-((-2*alpha*(z-x)-beta)*exp(-alpha*(z-x)^2-beta*(z-x))*
  (-exp(-(x-delta)^4)-exp(-(x+delta)^2)+(exp(-(y-delta)^4)+exp(-
  (y+delta)^2))*exp(-alpha*(x-y)^2-beta*(x-y)))/(-1+exp(-alpha*(x-
  y)^2-beta*(x-y))*exp(-alpha*(y-x)^2-beta*(y-x)))-(-2*alpha*(z-y)
  -beta)*exp(-alpha*(z-y)^2-beta*(z-y))/(-1+exp(-alpha*(x-y)^2-
  beta*(x-y))*exp(-alpha*(y-x)^2-beta*(y-x)))*(-(exp(-(x-delta)^4)
  +exp(-(x+delta)^2))*exp(-alpha*(y-x)^2-beta*(y-x))+exp(-(y-
  delta)^4)+exp(-(y+delta)^2)))/(exp(-(z-delta)^4)+exp(-(z+delta)
  ^2))+(exp(-alpha*(z-x)^2-beta*(z-x))*(-exp(-(x-delta)^4)-exp(-
  (x+delta)^2)+(exp(-(y-delta)^4)+exp(-(y+delta)^2))*exp(-alpha*
  (x-y)^2-beta*(x-y)))/(-1+exp(-alpha*(x-y)^2-beta*(x-y))*exp(-
  alpha*(y-x)^2-beta*(y-x)))-exp(-alpha*(z-y)^2-beta*(z-y))/(-1+
  exp(-alpha*(x-y)^2-beta*(x-y))*exp(-alpha*(y-x)^2-beta*(y-x)))*
  (-(exp(-(x-delta)^4)+exp(-(x+delta)^2))*exp(-alpha*(y-x)^2-beta*
  (y-x))+exp(-(y-delta)^4)+exp(-(y+delta)^2)))/(exp(-(z-delta)^4)+
  exp(-(z+delta)^2))^2*(-4*(z-delta)^3*exp(-(z-delta)^4)+(-2*z-2*
  delta)*exp(-(z+delta)^2)));
```

$$-\frac{1}{e^{-(y-d)^4} + e^{-(y+d)^2}} \left(\frac{1}{-1 + e^{-a (x-y)^2 - b (x-y)}\, e^{-a (y-x)^2 - b (y-x)}} \left(\left(-2 a (y-x)\right. \right. \right. \tag{26}$$

$$K\,b)\,e^{-a\,(y-x)^2-b\,(y-x)}\left(-e^{-(x-d)^4}-e^{-(x+d)^2}+\left(e^{-(y-d)^4}+e^{-(y+d)^2}\right)e^{-a\,(x-y)^2-b\,(x-y)}\right)\Bigg)$$

$$+\frac{b\,e^{0}\left(-\left(e^{-(x-d)^4}+e^{-(x+d)^2}\right)e^{-a\,(y-x)^2-b\,(y-x)}+e^{-(y-d)^4}+e^{-(y+d)^2}\right)}{-1+e^{-a\,(x-y)^2-b\,(x-y)}\,e^{-a\,(y-x)^2-b\,(y-x)}}\Bigg)$$

$$+\frac{1}{\left(e^{-(y-d)^4}+e^{-(y+d)^2}\right)^2}\Bigg(\Bigg(1\Big/\Big(-1$$

$$+e^{-a\,(x-y)^2-b\,(x-y)}\,e^{-a\,(y-x)^2-b\,(y-x)}\Big)\Big(e^{-a\,(y-x)^2-b\,(y-x)}\left(-e^{-(x-d)^4}-e^{-(x+d)^2}\right.$$

$$\left.+\left(e^{-(y-d)^4}+e^{-(y+d)^2}\right)e^{-a\,(x-y)^2-b\,(x-y)}\right)\Bigg)$$

$$-\frac{e^{0}\left(-\left(e^{-(x-d)^4}+e^{-(x+d)^2}\right)e^{-a\,(y-x)^2-b\,(y-x)}+e^{-(y-d)^4}+e^{-(y+d)^2}\right)}{-1+e^{-a\,(x-y)^2-b\,(x-y)}\,e^{-a\,(y-x)^2-b\,(y-x)}}\Bigg)\Bigg($$

$$-4\,(y-d)^3\,e^{-(y-d)^4}+\left(-2\,y+2\,d\right)e^{-(y+d)^2}\Bigg)$$

```
Ds[y]:=(x,y)->-((-2*alpha*(y-x)-beta)*exp(-alpha*(y-x)^2-beta*
(y-x))*(-exp(-(x-delta)^4)-exp(-(x+delta)^2)+(exp(-(y-delta)^4)+
exp(-(y+delta)^2))*exp(-alpha*(x-y)^2-beta*(x-y)))/(-1+exp(-
alpha*(x-y)^2-beta*(x-y))*exp(-alpha*(y-x)^2-beta*(y-x)))+beta*
exp(0)/(-1+exp(-alpha*(x-y)^2-beta*(x-y))*exp(-alpha*(y-x)^2-
beta*(y-x)))*(-(exp(-(x-delta)^4)+exp(-(x+delta)^2))*exp(-alpha*
(y-x)^2-beta*(y-x))+exp(-(y-delta)^4)+exp(-(y+delta)^2)))/(exp(-
(y-delta)^4)+exp(-(y+delta)^2))+(exp(-alpha*(y-x)^2-beta*(y-x))*
(-exp(-(x-delta)^4)-exp(-(x+delta)^2)+(exp(-(y-delta)^4)+exp(-
(y+delta)^2))*exp(-alpha*(x-y)^2-beta*(x-y)))/(-1+exp(-alpha*(x-
y)^2-beta*(x-y))*exp(-alpha*(y-x)^2-beta*(y-x)))-exp(0)/(-1+exp
(-alpha*(x-y)^2-beta*(x-y))*exp(-alpha*(y-x)^2-beta*(y-x)))*(-
(exp(-(x-delta)^4)+exp(-(x+delta)^2))*exp(-alpha*(y-x)^2-beta*
(y-x))+exp(-(y-delta)^4)+exp(-(y+delta)^2)))/(exp(-(y-delta)^4)+
exp(-(y+delta)^2))^2*(-4*(y-delta)^3*exp(-(y-delta)^4)+(-2*y-2*
delta)*exp(-(y+delta)^2));
```

$$Ds_y := (x, y) \;/\; -\frac{1}{e^{-(y-d)^4} + e^{-(y+d)^2}} \left(\frac{1}{-1 + e^{-a(x-y)^2 - b(x-y)}\, e^{-a(y-x)^2 - b(y-x)}} \left(\left(-2a(y-x) - b \right) e^{-a(y-x)^2 - b(y-x)} \left(-e^{-(x-d)^4} - e^{-(x+d)^2} + \left(e^{-(y-d)^4} + e^{-(y+d)^2} \right) e^{-a(x-y)^2 - b(x-y)} \right) \right) + \frac{b e^0 \left(-\left(e^{-(x-d)^4} + e^{-(x+d)^2} \right) e^{-a(y-x)^2 - b(y-x)} + e^{-(y-d)^4} + e^{-(y+d)^2} \right)}{-1 + e^{-a(x-y)^2 - b(x-y)}\, e^{-a(y-x)^2 - b(y-x)}} \right) + \frac{1}{\left(e^{-(y-d)^4} + e^{-(y+d)^2} \right)^2} \left(\left(1 \Big/ \left(-1 + e^{-a(x-y)^2 - b(x-y)}\, e^{-a(y-x)^2 - b(y-x)} \right) \left(e^{-a(y-x)^2 - b(y-x)} \left(-e^{-(x-d)^4} - e^{-(x+d)^2} + \left(e^{-(y-d)^4} + e^{-(y+d)^2} \right) e^{-a(x-y)^2 - b(x-y)} \right) \right) \right) - \frac{e^0 \left(-\left(e^{-(x-d)^4} + e^{-(x+d)^2} \right) e^{-a(y-x)^2 - b(y-x)} + e^{-(y-d)^4} + e^{-(y+d)^2} \right)}{-1 + e^{-a(x-y)^2 - b(x-y)}\, e^{-a(y-x)^2 - b(y-x)}} \right) \left(-4(y-d)^3 e^{-(y-d)^4} + (-2y - 2d) e^{-(y+d)^2} \right) \right) \tag{27}$$

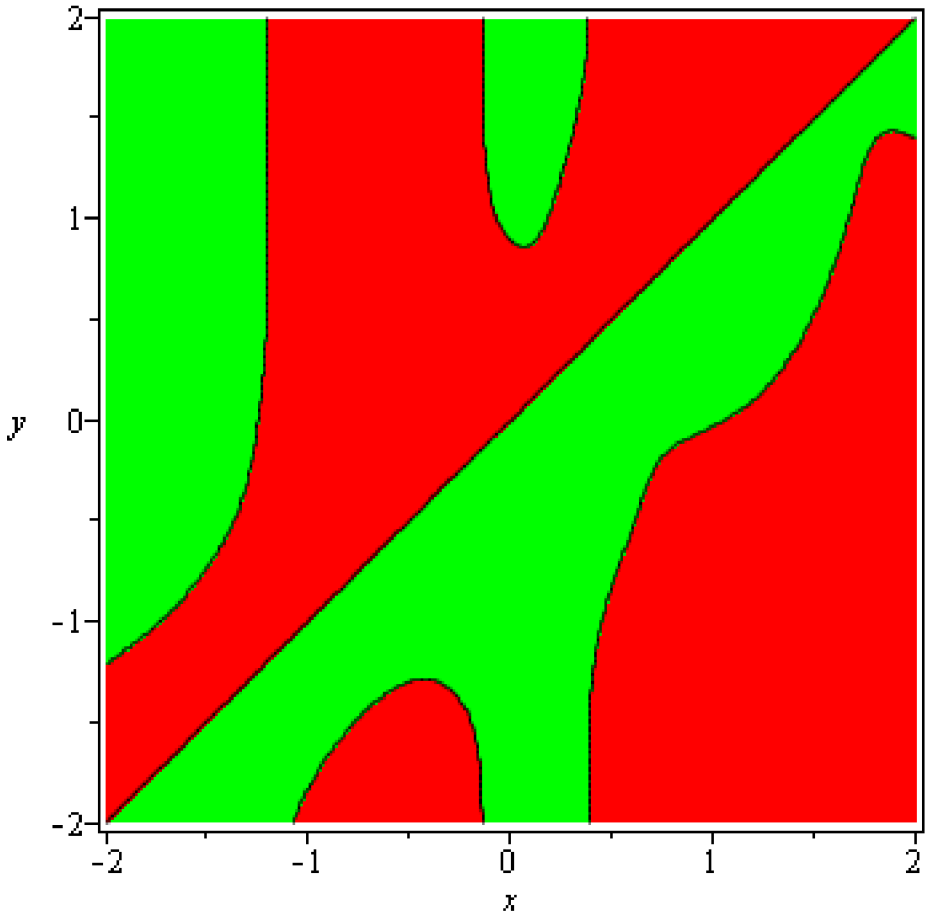
```
alpha:=2;beta:=-0.4;delta:=1;
```

$$a := 2$$

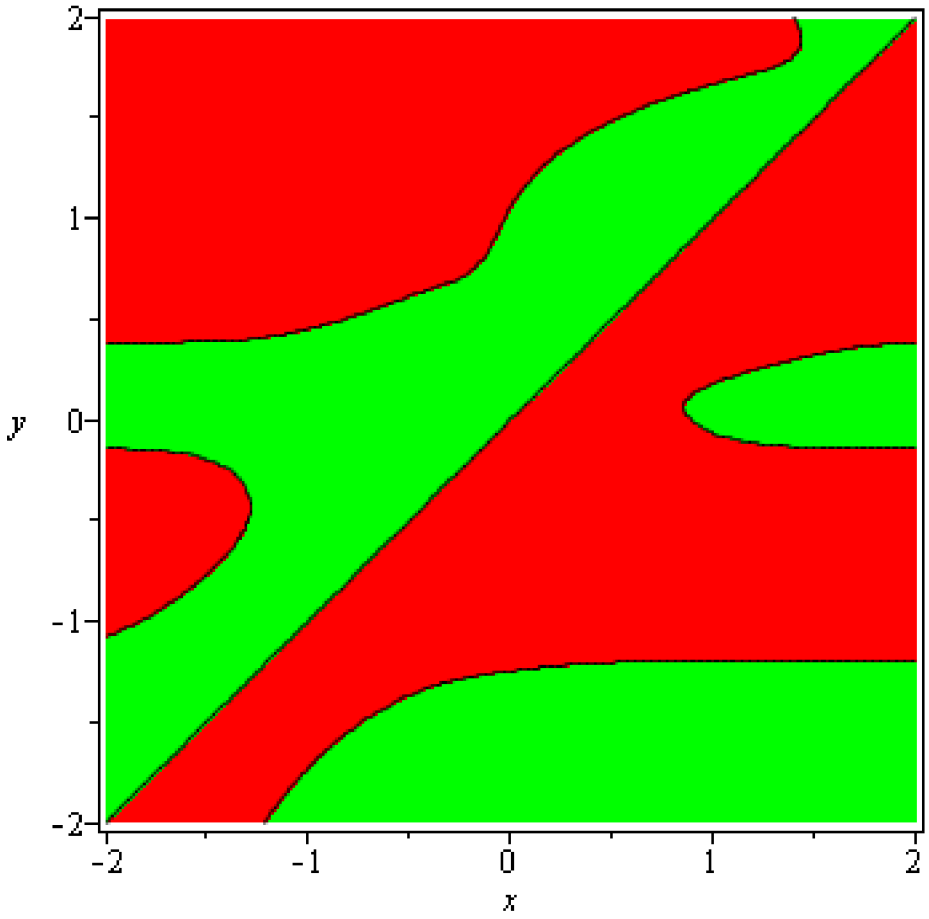
$$b := -0.4$$

$$d := 1 \tag{28}$$

```
with(plots):
contourplot(piecewise(abs(x-y)<0.01,0,Ds[x](x,y)),x=-2..2,y=-2.
.2,contours=[0],filled=true,coloring=[red,green],grid=[100,100],
axes=boxed);
```



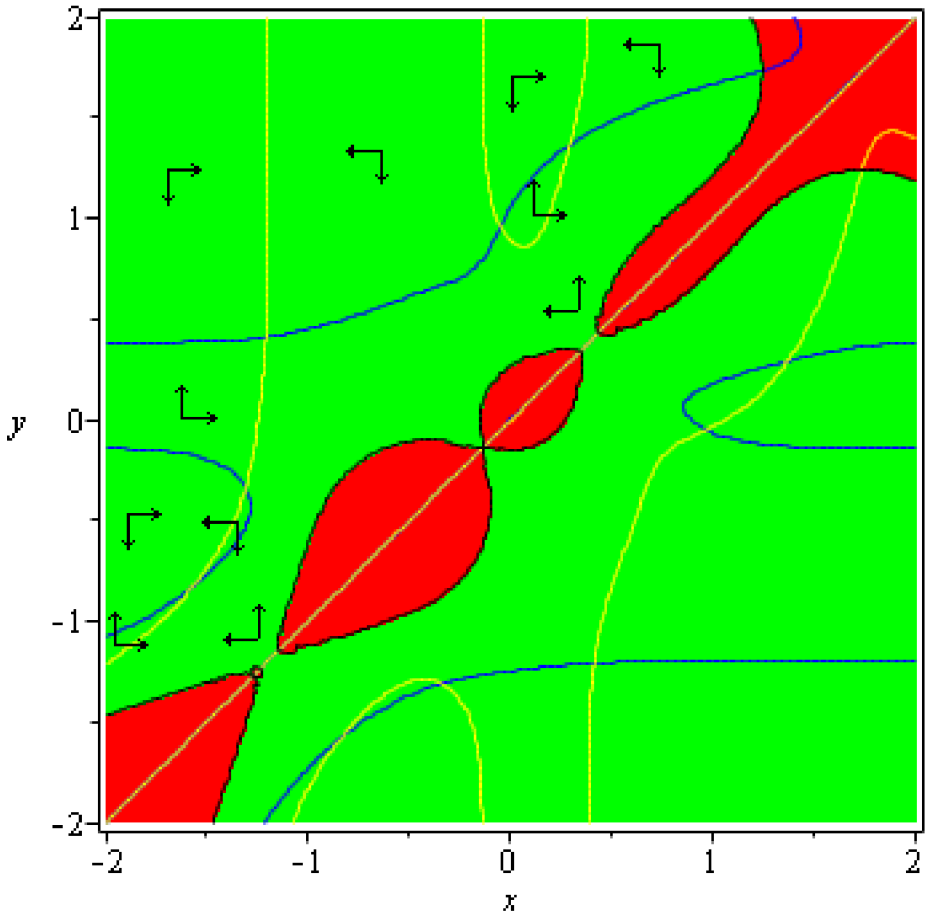

```
> contourplot(piecewise(abs(x-y)<0.01,0,Ds[y](x,y)),x=-2..2,y=-2.
  .2,contours=[0],filled=true,coloring=[red,green],grid=[100,100],
  axes=boxed);
```




```
> A:=contourplot(piecewise(abs(x-y)<0.01,0,Ds[x](x,y)),x=-2..2,y=
  -2..2,contours=[0],color=yellow,grid=[100,100],axes=boxed):
> B:=contourplot(piecewise(abs(x-y)<0.01,0,Ds[y](x,y)),x=-2..2,y=
  -2..2,contours=[0],color=blue,grid=[100,100],axes=boxed):
> s_mono:=(x,y)->r(y)*(1-(a(y,x)*K(x))/K(y));
```

$$s\_mono := (x, y) \;/\; r(y) \left(1 \;\mathrm{K}\; \frac{a(y, x)\, K(x)}{K(y)} \right) \qquad (29)$$

```
> C:=contourplot(s_mono(x,y)*s_mono(y,x),x=-2..2,y=-2..2,contours=
  [0], filled=true, coloring=[red,green], grid=[100,100],axes=
  boxed):
> display({A,B,C});
```




```
O subs(z=x,diff(s(x,y,z),z$2));
```

$$
\begin{aligned}
&-\frac{1}{e^{-(x-1)^4}+e^{-(x+1)^2}}\left(\right. \qquad \textbf{(30)}\\
&\quad -\frac{3.84\,e^{0.}\left(-1.\,e^{-(x-1)^4}-1.\,e^{-(x+1)^2}+\left(e^{-(y-1)^4}+e^{-(y+1)^2}\right)e^{-2(x-y)^2+0.4x-0.4y}\right)}{-1.+e^{-2(x-y)^2+0.4x-0.4y}\,e^{-2(y-x)^2+0.4y-0.4x}}\\
&\quad +\left(4\,e^{-2(x-y)^2+0.4x-0.4y}\left(-\left(e^{-(x-1)^4}+e^{-(x+1)^2}\right)e^{-2(y-x)^2+0.4y-0.4x}\right.\right.\\
&\quad \left.\left.+1.\,e^{-(y-1)^4}+1.\,e^{-(y+1)^2}\right)\right)\Big/\left(-1.+e^{-2(x-y)^2+0.4x-0.4y}\,e^{-2(y-x)^2+0.4y-0.4x}\right)\\
&\quad -\left(\left(-4x+4y+0.4\right)^2e^{-2(x-y)^2+0.4x-0.4y}\left(-\left(e^{-(x-1)^4}\right.\right.\right.\\
&\quad \left.\left.\left.+e^{-(x+1)^2}\right)e^{-2(y-x)^2+0.4y-0.4x}+1.\,e^{-(y-1)^4}+1.\,e^{-(y+1)^2}\right)\right)\Big/\left(-1.\right.\\
&\quad \left.+e^{-2(x-y)^2+0.4x-0.4y}\,e^{-2(y-x)^2+0.4y-0.4x}\right)\Big)\\
&\quad +\frac{1}{\left(e^{-(x-1)^4}+e^{-(x+1)^2}\right)^2}\left(2\left(\left(0.4\,e^{0.}\left(-1.\,e^{-(x-1)^4}-1.\,e^{-(x+1)^2}+\left(e^{-(y-1)^4}\right.\right.\right.\right.
\end{aligned}
$$

$$+ e^{-2(y+1)^2}\Big)\, e^{-2(x-y)^2+0.4x-0.4y}\Big)\Big)\Big/\Big(-1.$$

$$+ e^{-2(x-y)^2+0.4x-0.4y}\, e^{-2(y-x)^2+0.4y-0.4x}\Big)\Big(-4(x-1)^3 e^{-(x-1)^4} + (-2x-2)\, e^{-(x+1)^2}\Big)\Big)$$

$$- \frac{1}{\left(e^{-(x-1)^4}+e^{-(x+1)^2}\right)^3}\left(2\left(\left(e^{0.}\left(-1.\, e^{-(x-1)^4} - 1.\, e^{-(x+1)^2} + \left(e^{-(y-1)^4}\right.\right.\right.\right.$$

$$+ e^{-2(y+1)^2}\Big)\, e^{-2(x-y)^2+0.4x-0.4y}\Big)\Big)\Big/\Big(-1.$$

$$+ e^{-2(x-y)^2+0.4x-0.4y}\, e^{-2(y-x)^2+0.4y-0.4x}\Big)\Big)\Big(-4(x-1)^3 e^{-(x-1)^4} + \Big(-2x$$

$$- 2\Big)\, e^{-(x+1)^2}\Big)^2\Big)$$

$$+ \frac{1}{\left(e^{-(x-1)^4}+e^{-(x+1)^2}\right)^2}\left(\left(\left(e^{0.}\left(-1.\, e^{-(x-1)^4} - 1.\, e^{-(x+1)^2} + \left(e^{-(y-1)^4}\right.\right.\right.\right.$$

$$+ e^{-2(y+1)^2}\Big)\, e^{-2(x-y)^2+0.4x-0.4y}\Big)\Big)\Big/\Big(-1.$$

$$+ e^{-2(x-y)^2+0.4x-0.4y}\, e^{-2(y-x)^2+0.4y-0.4x}\Big)\Big)\Big(-12(x-1)^2 e^{-(x-1)^4} + 16(x$$

$$- 1)^6 e^{-(x-1)^4} - 2e^{-(x+1)^2} + \Big(-2x-2\Big)^2 e^{-(x+1)^2}\Big)\Big)$$

```
D2s[x]:=(x,y)->-(-3.84*exp(0.)*(-1.*exp(-(x-1)^4)-1.*exp(-(x+1)
^2)+(exp(-(y-1)^4)+exp(-(y+1)^2))*exp(-2*(x-y)^2+.4*x-.4*y))/
(-1.+exp(-2*(x-y)^2+.4*x-.4*y)*exp(-2*(y-x)^2+.4*y-.4*x))+4*exp
(-2*(x-y)^2+.4*x-.4*y)/(-1.+exp(-2*(x-y)^2+.4*x-.4*y)*exp(-2*(y-
x)^2+.4*y-.4*x))*(-(exp(-(x-1)^4)+exp(-(x+1)^2))*exp(-2*(y-x)
^2+.4*y-.4*x)+1.*exp(-(y-1)^4)+1.*exp(-(y+1)^2))-(-4*x+4*y+.4)
^2*exp(-2*(x-y)^2+.4*x-.4*y)/(-1.+exp(-2*(x-y)^2+.4*x-.4*y)*exp
(-2*(y-x)^2+.4*y-.4*x))*(-(exp(-(x-1)^4)+exp(-(x+1)^2))*exp(-2*
```

```
(y-x)^2+.4*y-.4*x)+1.*exp(-(y-1)^4)+1.*exp(-(y+1)^2)))/(exp(-
(x-1)^4)+exp(-(x+1)^2))+2*(.4*exp(0.)*(-1.*exp(-(x-1)^4)-1.*exp
(-(x+1)^2)+(exp(-(y-1)^4)+exp(-(y+1)^2))*exp(-2*(x-y)^2+.4*x-.4*
y))/(-1.+exp(-2*(x-y)^2+.4*x-.4*y)*exp(-2*(y-x)^2+.4*y-.4*x))-
(-4*x+4*y+.4)*exp(-2*(x-y)^2+.4*x-.4*y)/(-1.+exp(-2*(x-y)^2+.4*
x-.4*y)*exp(-2*(y-x)^2+.4*y-.4*x))*(-(exp(-(x-1)^4)+exp(-(x+1)
^2))*exp(-2*(y-x)^2+.4*y-.4*x)+1.*exp(-(y-1)^4)+1.*exp(-(y+1)^2)
))/(exp(-(x-1)^4)+exp(-(x+1)^2))^2*(-4*(x-1)^3*exp(-(x-1)^4)+
(-2*x-2)*exp(-(x+1)^2))-2*(exp(0.)*(-1.*exp(-(x-1)^4)-1.*exp(-
(x+1)^2)+(exp(-(y-1)^4)+exp(-(y+1)^2))*exp(-2*(x-y)^2+.4*x-.4*y)
)/(-1.+exp(-2*(x-y)^2+.4*x-.4*y)*exp(-2*(y-x)^2+.4*y-.4*x))-exp
(-2*(x-y)^2+.4*x-.4*y)/(-1.+exp(-2*(x-y)^2+.4*x-.4*y)*exp(-2*(y-
x)^2+.4*y-.4*x))*(-(exp(-(x-1)^4)+exp(-(x+1)^2))*exp(-2*(y-x)
^2+.4*y-.4*x)+1.*exp(-(y-1)^4)+1.*exp(-(y+1)^2)))/(exp(-(x-1)^4)
+exp(-(x+1)^2))^3*(-4*(x-1)^3*exp(-(x-1)^4)+(-2*x-2)*exp(-(x+1)
^2))^2+(exp(0.)*(-1.*exp(-(x-1)^4)-1.*exp(-(x+1)^2)+(exp(-(y-1)
^4)+exp(-(y+1)^2))*exp(-2*(x-y)^2+.4*x-.4*y))/(-1.+exp(-2*(x-y)
^2+.4*x-.4*y)*exp(-2*(y-x)^2+.4*y-.4*x))-exp(-2*(x-y)^2+.4*x-.4*
y)/(-1.+exp(-2*(x-y)^2+.4*x-.4*y)*exp(-2*(y-x)^2+.4*y-.4*x))*(-
(exp(-(x-1)^4)+exp(-(x+1)^2))*exp(-2*(y-x)^2+.4*y-.4*x)+1.*exp(-
(y-1)^4)+1.*exp(-(y+1)^2)))/(exp(-(x-1)^4)+exp(-(x+1)^2))^2*
(-12*(x-1)^2*exp(-(x-1)^4)+16*(x-1)^6*exp(-(x-1)^4)-2*exp(-(x+1)
^2)+(-2*x-2)^2*exp(-(x+1)^2));
```

$$D2s_x := (x, y) \;/\; \left| \frac{1}{e^{\left|(x-1)^4\right.} + e^{\left|(x+1)^2\right.}} \left(\right.\right. \tag{31}$$

$$\left| \frac{3.84\, e^{0.} \left(\left| 1.\, e^{\left|(x-1)^4\right.} - 1.\, e^{\left|(x+1)^2\right.} + \left(e^{\left|(y-1)^4\right.} + e^{\left|(y+1)^2\right.} \right) e^{\left|2\,(x-y)^2 + 0.4\,x - 0.4\,y\right.} \right)}{\left| 1.\, e^{\left|2\,(x-y)^2 + 0.4\,x - 0.4\,y\right.} e^{\left|2\,(y-x)^2 + 0.4\,y - 0.4\,x\right.}} \right.$$

$$+ \left(4\, e^{\left|2\,(x-y)^2 + 0.4\,x - 0.4\,y\right.} \left(\left| \left(e^{\left|(x-1)^4\right.} + e^{\left|(x+1)^2\right.} \right) e^{\left|2\,(y-x)^2 + 0.4\,y - 0.4\,x\right.} \right.\right.\right.$$

$$\left.\left.\left. + 1.\, e^{\left|(y-1)^4\right.} + 1.\, e^{\left|(y+1)^2\right.} \right) \right) \Big/ \left(\left| 1.\, e^{\left|2\,(x-y)^2 + 0.4\,x - 0.4\,y\right.} e^{\left|2\,(y-x)^2 + 0.4\,y - 0.4\,x\right.} \right) \right.$$

$$- \left(\left(\left| 4\,x + 4\,y + 0.4 \right.\right)^2 e^{\left|2\,(x-y)^2 + 0.4\,x - 0.4\,y\right.} \left(\left| \left(e^{\left|(x-1)^4\right.} \right.\right.\right.\right.$$

$$\left.\left.\left.\left. + e^{\left|(x+1)^2\right.} \right) e^{\left|2\,(y-x)^2 + 0.4\,y - 0.4\,x\right.} + 1.\, e^{\left|(y-1)^4\right.} + 1.\, e^{\left|(y+1)^2\right.} \right) \right) \Big/ \left(\left| 1. \right.\right.\right.$$

$$\left.\left. + e^{\left|2\,(x-y)^2 + 0.4\,x - 0.4\,y\right.} e^{\left|2\,(y-x)^2 + 0.4\,y - 0.4\,x\right.} \right) \right)$$

$$+ \frac{1}{\left(e^{\left|(x-1)^4\right.} + e^{\left|(x+1)^2\right.} \right)^2} \left(2 \left(\left(0.4\, e^{0.} \left(\left| 1.\, e^{\left|(x-1)^4\right.} - 1.\, e^{\left|(x+1)^2\right.} + \left(e^{\left|(y-1)^4\right.} \right.\right.\right.\right.\right.$$

$$+ e^{-(y+1)^2}\right) e^{-2(x-y)^2+0.4x-0.4y}\right)\Big/\left(-1.\right.$$

$$+ e^{-2(x-y)^2+0.4x-0.4y} e^{-2(y-x)^2+0.4y-0.4x}\right) - \left(\left(-4x+4y\right.\right.$$
$$\left.+0.4\right) e^{-2(x-y)^2+0.4x-0.4y}\left(-\left(e^{-(x-1)^4}+e^{-(x+1)^2}\right) e^{-2(y-x)^2+0.4y-0.4x}\right.$$
$$\left.+1.\, e^{-(y-1)^4}+1.\, e^{-(y+1)^2}\right)\Big)\Big/\left(-1.+e^{-2(x-y)^2+0.4x-0.4y} e^{-2(y-x)^2+0.4y-0.4x}\right)\Big)\Big($$
$$-4(x-1)^3 e^{-(x-1)^4}+(-2x-2) e^{-(x+1)^2}\Big)\Big)$$
$$-\frac{1}{\left(e^{-(x-1)^4}+e^{-(x+1)^2}\right)^3}\left(2\left(\left(e^{0.}\left(-1.\, e^{-(x-1)^4}-1.\, e^{-(x+1)^2}+\left(e^{-(y-1)^4}\right.\right.\right.\right.$$
$$\left.\left.\left.\left.+e^{-(y+1)^2}\right) e^{-2(x-y)^2+0.4x-0.4y}\right)\Big)\Big/\left(-1.\right.\right.$$

$$+ e^{-2(x-y)^2+0.4x-0.4y} e^{-2(y-x)^2+0.4y-0.4x}\right)\left(e^{-2(x-y)^2+0.4x-0.4y}\left(-\left(e^{-(x-1)^4}\right.\right.\right.$$
$$\left.\left.\left.+e^{-(x+1)^2}\right) e^{-2(y-x)^2+0.4y-0.4x}+1.\, e^{-(y-1)^4}+1.\, e^{-(y+1)^2}\right)\Big)\Big/\left(-1.\right.$$
$$\left.+ e^{-2(x-y)^2+0.4x-0.4y} e^{-2(y-x)^2+0.4y-0.4x}\right)\Big)\left(-4(x-1)^3 e^{-(x-1)^4}+(-2x\right.$$
$$\left.-2) e^{-(x+1)^2}\right)^2\Big)$$
$$+\frac{1}{\left(e^{-(x-1)^4}+e^{-(x+1)^2}\right)^2}\left(\left(\left(e^{0.}\left(-1.\, e^{-(x-1)^4}-1.\, e^{-(x+1)^2}+\left(e^{-(y-1)^4}\right.\right.\right.\right.$$
$$\left.\left.\left.\left.+e^{-(y+1)^2}\right) e^{-2(x-y)^2+0.4x-0.4y}\right)\Big)\Big/\left(-1.\right.\right.$$

$$+ e^{-2(x-y)^2+0.4x-0.4y} e^{-2(y-x)^2+0.4y-0.4x}\right)\left(e^{-2(x-y)^2+0.4x-0.4y}\left(-\left(e^{-(x-1)^4}\right.\right.\right.$$
$$\left.\left.\left.+e^{-(x+1)^2}\right) e^{-2(y-x)^2+0.4y-0.4x}+1.\, e^{-(y-1)^4}+1.\, e^{-(y+1)^2}\right)\Big)\Big/\left(-1.\right.$$
$$\left.+ e^{-2(x-y)^2+0.4x-0.4y} e^{-2(y-x)^2+0.4y-0.4x}\right)\Big)\left(-12(x-1)^2 e^{-(x-1)^4}+16(x\right.$$
$$\left.-1)^6 e^{-(x-1)^4}-2 e^{-(x+1)^2}+(-2x-2)^2 e^{-(x+1)^2}\right)\Big)$$

```
O subs(z=y,diff(s(x,y,z),z$2));
```

$$-\frac{1}{e^{-(y-1)^4}+e^{-(y+1)^2}}\left(-\left(4 e^{-2(y-x)^2+0.4y-0.4x}\left(-1.\, e^{-(x-1)^4}-1.\, e^{-(x+1)^2}+\left(e^{-(y-1)^4}\right.\right.\right.\right.$$
$$\left.\left.\left.\left.+e^{-(y+1)^2}\right) e^{-2(x-y)^2+0.4x-0.4y}\right)\Big)\Big/\left(-1.\right.\right.$$

$$\left.\left.+ e^{-2(x-y)^2+0.4x-0.4y} e^{-2(y-x)^2+0.4y-0.4x}\right)\right.$$

$$+ e^{-2 (x-y)^2 + 0.4 x - 0.4 y} e^{-2 (y-x)^2 + 0.4 y - 0.4 x}\Big) + \Big((-4 y + 4 x + 0.4)^2 e^{-2 (y-x)^2 + 0.4 y - 0.4 x} \Big(-1. e^{-(x-1)^4}$$

$$-1. + e^{-2 (x-y)^2 + 0.4 x - 0.4 y} e^{-2 (y-x)^2 + 0.4 y - 0.4 x}\Big)$$

$$+ \Big(3.84 e^{0.} \Big(-\Big(e^{-(x-1)^4} + e^{-(x+1)^2}\Big) e^{-2 (y-x)^2 + 0.4 y - 0.4 x} + 1. e^{-(y-1)^4}$$
$$+ 1. e^{-(y+1)^2}\Big)\Big) \Big/ \Big(-1. + e^{-2 (x-y)^2 + 0.4 x - 0.4 y} e^{-2 (y-x)^2 + 0.4 y - 0.4 x}\Big)\Big)$$

$$+ \frac{1}{\left(e^{-(y-1)^4} + e^{-(y+1)^2}\right)^2} \Bigg(2 \Bigg(\Big((-4 y + 4 x + 0.4) e^{-2 (y-x)^2 + 0.4 y - 0.4 x} \Big($$

$$-1. e^{-(x-1)^4} - 1. e^{-(x+1)^2} + \Big(e^{-(y-1)^4} + e^{-(y+1)^2}\Big) e^{-2 (x-y)^2 + 0.4 x - 0.4 y}\Big)\Big) \Big/ \Big(-1.$$

$$+ e^{-2 (x-y)^2 + 0.4 x - 0.4 y} e^{-2 (y-x)^2 + 0.4 y - 0.4 x}\Big)$$

$$- \frac{0.4 e^{0.} \Big(-\Big(e^{-(x-1)^4} + e^{-(x+1)^2}\Big) e^{-2 (y-x)^2 + 0.4 y - 0.4 x} + 1. e^{-(y-1)^4} + 1. e^{-(y+1)^2}\Big)}{-1. + e^{-2 (x-y)^2 + 0.4 x - 0.4 y} e^{-2 (y-x)^2 + 0.4 y - 0.4 x}} \Bigg)$$

$$\Big(-4 (y-1)^3 e^{-(y-1)^4} + (-2 y + 2) e^{-(y+1)^2}\Big) \Bigg)$$

$$- \frac{1}{\left(e^{-(y-1)^4} + e^{-(y+1)^2}\right)^3} \Bigg(2 \Bigg(\Big(e^{-2 (y-x)^2 + 0.4 y - 0.4 x} \Big(-1. e^{-(x-1)^4} - 1. e^{-(x+1)^2}$$

$$+ \Big(e^{-(y-1)^4} + e^{-(y+1)^2}\Big) e^{-2 (x-y)^2 + 0.4 x - 0.4 y}\Big)\Big) \Big/ \Big(-1.$$

$$+ e^{-2 (x-y)^2 + 0.4 x - 0.4 y} e^{-2 (y-x)^2 + 0.4 y - 0.4 x}\Big)$$

$$- \frac{e^{0.} \Big(-\Big(e^{-(x-1)^4} + e^{-(x+1)^2}\Big) e^{-2 (y-x)^2 + 0.4 y - 0.4 x} + 1. e^{-(y-1)^4} + 1. e^{-(y+1)^2}\Big)}{-1. + e^{-2 (x-y)^2 + 0.4 x - 0.4 y} e^{-2 (y-x)^2 + 0.4 y - 0.4 x}} \Bigg)$$

$$\Big(-4 (y-1)^3 e^{-(y-1)^4} + (-2 y + 2) e^{-(y+1)^2}\Big)^2 \Bigg)$$

$$+ \frac{1}{\left(e^{-(y-1)^4} + e^{-(y+1)^2}\right)^2} \Bigg(\Bigg(\Big(e^{-2 (y-x)^2 + 0.4 y - 0.4 x} \Big(-1. e^{-(x-1)^4} - 1. e^{-(x+1)^2}$$

$$C\left(e^{-(y-1)^4}+e^{-(y+1)^2}\right)e^{-2(x-y)^2+0.4x-0.4y}\Big)\Big/\Big(-1.$$

$$+e^{-2(x-y)^2+0.4x-0.4y}e^{-2(y-x)^2+0.4y-0.4x}\Big)$$

$$-\frac{e^{0.}\left(-\left(e^{-(x-1)^4}+e^{-(x+1)^2}\right)e^{-2(y-x)^2+0.4y-0.4x}+1.e^{-(y-1)^4}+1.e^{-(y+1)^2}\right)}{-1.+e^{-2(x-y)^2+0.4x-0.4y}e^{-2(y-x)^2+0.4y-0.4x}}\Bigg)\Bigg($$

$$-12(y-1)^2e^{-(y-1)^4}+16(y-1)^6e^{-(y-1)^4}-2e^{-(y+1)^2}+(-2y-2)^2e^{-(y+1)^2}\Bigg)$$

```
D2s[y]:=(x,y)->-(-4*exp(-2*(y-x)^2+.4*y-.4*x)*(-1.*exp(-(x-1)^4)
-1.*exp(-(x+1)^2)+(exp(-(y-1)^4)+exp(-(y+1)^2))*exp(-2*(x-y)
^2+.4*x-.4*y))/(-1.+exp(-2*(x-y)^2+.4*x-.4*y)*exp(-2*(y-x)^2+.4*
y-.4*x))+(-4*y+4*x+.4)^2*exp(-2*(y-x)^2+.4*y-.4*x)*(-1.*exp(-
(x-1)^4)-1.*exp(-(x+1)^2)+(exp(-(y-1)^4)+exp(-(y+1)^2))*exp(-2*
(x-y)^2+.4*x-.4*y))/(-1.+exp(-2*(x-y)^2+.4*x-.4*y)*exp(-2*(y-x)
^2+.4*y-.4*x))+3.84*exp(0.)/(-1.+exp(-2*(x-y)^2+.4*x-.4*y)*exp
(-2*(y-x)^2+.4*y-.4*x))*(-(exp(-(x-1)^4)+exp(-(x+1)^2))*exp(-2*
(y-x)^2+.4*y-.4*x)+1.*exp(-(y-1)^4)+1.*exp(-(y+1)^2)))/(exp(-
(y-1)^4)+exp(-(y+1)^2))+2*((-4*y+4*x+.4)*exp(-2*(y-x)^2+.4*y-.4*
x)*(-1.*exp(-(x-1)^4)-1.*exp(-(x+1)^2)+(exp(-(y-1)^4)+exp(-(y+1)
^2))*exp(-2*(x-y)^2+.4*x-.4*y))/(-1.+exp(-2*(x-y)^2+.4*x-.4*y)*
exp(-2*(y-x)^2+.4*y-.4*x))-.4*exp(0.)/(-1.+exp(-2*(x-y)^2+.4*
x-.4*y)*exp(-2*(y-x)^2+.4*y-.4*x))*(-(exp(-(x-1)^4)+exp(-(x+1)
^2))*exp(-2*(y-x)^2+.4*y-.4*x)+1.*exp(-(y-1)^4)+1.*exp(-(y+1)^2)
))/(exp(-(y-1)^4)+exp(-(y+1)^2))^2*(-4*(y-1)^3*exp(-(y-1)^4)+
(-2*y-2)*exp(-(y+1)^2))-2*(exp(-2*(y-x)^2+.4*y-.4*x)*(-1.*exp(-
(x-1)^4)-1.*exp(-(x+1)^2)+(exp(-(y-1)^4)+exp(-(y+1)^2))*exp(-2*
(x-y)^2+.4*x-.4*y))/(-1.+exp(-2*(x-y)^2+.4*x-.4*y)*exp(-2*(y-x)
^2+.4*y-.4*x))-exp(0.)/(-1.+exp(-2*(x-y)^2+.4*x-.4*y)*exp(-2*(y-
x)^2+.4*y-.4*x))*(-(exp(-(x-1)^4)+exp(-(x+1)^2))*exp(-2*(y-x)
^2+.4*y-.4*x)+1.*exp(-(y-1)^4)+1.*exp(-(y+1)^2)))/(exp(-(y-1)^4)
+exp(-(y+1)^2))^3*(-4*(y-1)^3*exp(-(y-1)^4)+(-2*y-2)*exp(-(y+1)
^2))^2+(exp(-2*(y-x)^2+.4*y-.4*x)*(-1.*exp(-(x-1)^4)-1.*exp(-
(x+1)^2)+(exp(-(y-1)^4)+exp(-(y+1)^2))*exp(-2*(x-y)^2+.4*x-.4*y)
)/(-1.+exp(-2*(x-y)^2+.4*x-.4*y)*exp(-2*(y-x)^2+.4*y-.4*x))-exp
(0.)/(-1.+exp(-2*(x-y)^2+.4*x-.4*y)*exp(-2*(y-x)^2+.4*y-.4*x))*
(-(exp(-(x-1)^4)+exp(-(x+1)^2))*exp(-2*(y-x)^2+.4*y-.4*x)+1.*exp
(-(y-1)^4)+1.*exp(-(y+1)^2)))/(exp(-(y-1)^4)+exp(-(y+1)^2))^2*
(-12*(y-1)^2*exp(-(y-1)^4)+16*(y-1)^6*exp(-(y-1)^4)-2*exp(-(y+1)
^2)+(-2*y-2)^2*exp(-(y+1)^2));
```

$$D2s_y := (x, y) \;/\; -\frac{1}{e^{-(y-1)^4} + e^{-(y+1)^2}} \left(-\left(4\, e^{-2(y-x)^2 + 0.4y - 0.4x} \left(-1.\, e^{-(x-1)^4} - 1.\, e^{-(x+1)^2} + \left(e^{-(y-1)^4} + e^{-(y+1)^2} \right) e^{-2(x-y)^2 + 0.4x - 0.4y} \right) \right) \Big/ \left(-1. + e^{-2(x-y)^2 + 0.4x - 0.4y} + e^{-2(y-x)^2 + 0.4y - 0.4x} \right) + \left((-4y + 4x + 0.4)^2 e^{-2(y-x)^2 + 0.4y - 0.4x} \left(-1.\, e^{-(x-1)^4} - 1.\, e^{-(x+1)^2} + \left(e^{-(y-1)^4} + e^{-(y+1)^2} \right) e^{-2(x-y)^2 + 0.4x - 0.4y} \right) \right) \Big/ \left(-1. + e^{-2(x-y)^2 + 0.4x - 0.4y} + e^{-2(y-x)^2 + 0.4y - 0.4x} \right)^2 + \left(3.84\, e^{0.} \left(-\left(e^{-(x-1)^4} + e^{-(x+1)^2} \right) e^{-2(y-x)^2 + 0.4y - 0.4x} - 1.\, e^{-(y-1)^4} - 1.\, e^{-(y+1)^2} \right) \right) \Big/ \left(-1. + e^{-2(x-y)^2 + 0.4x - 0.4y} + e^{-2(y-x)^2 + 0.4y - 0.4x} \right) \right)$$

$$+ \frac{1}{\left(e^{-(y-1)^4} + e^{-(y+1)^2} \right)^2} \left(2 \left(\left((-4y + 4x + 0.4)\, e^{-2(y-x)^2 + 0.4y - 0.4x} \left(-1.\, e^{-(x-1)^4} - 1.\, e^{-(x+1)^2} + \left(e^{-(y-1)^4} + e^{-(y+1)^2} \right) e^{-2(x-y)^2 + 0.4x - 0.4y} \right) \right) \Big/ \left(-1. + e^{-2(x-y)^2 + 0.4x - 0.4y} + e^{-2(y-x)^2 + 0.4y - 0.4x} \right) \right) \right.$$

$$\left. - \frac{0.4\, e^{0.} \left(-\left(e^{-(x-1)^4} + e^{-(x+1)^2} \right) e^{-2(y-x)^2 + 0.4y - 0.4x} - 1.\, e^{-(y-1)^4} - 1.\, e^{-(y+1)^2} \right)}{-1. + e^{-2(x-y)^2 + 0.4x - 0.4y} + e^{-2(y-x)^2 + 0.4y - 0.4x}} \right)$$

$$\left(-4\,(y-1)^3 e^{-(y-1)^4} + (-2y + 2)\, e^{-(y+1)^2} \right) \Bigg)$$

$$- \frac{1}{\left(e^{-(y-1)^4} + e^{-(y+1)^2} \right)^3} \left(2 \left(\left(e^{-2(y-x)^2 + 0.4y - 0.4x} \left(-1.\, e^{-(x-1)^4} - 1.\, e^{-(x+1)^2} + \left(e^{-(y-1)^4} + e^{-(y+1)^2} \right) e^{-2(x-y)^2 + 0.4x - 0.4y} \right) \right) \Big/ \left(-1. + e^{-2(x-y)^2 + 0.4x - 0.4y} + e^{-2(y-x)^2 + 0.4y - 0.4x} \right) \right) \right.$$

$$\left. - \frac{e^{0.} \left(-\left(e^{-(x-1)^4} + e^{-(x+1)^2} \right) e^{-2(y-x)^2 + 0.4y - 0.4x} - 1.\, e^{-(y-1)^4} - 1.\, e^{-(y+1)^2} \right)}{-1. + e^{-2(x-y)^2 + 0.4x - 0.4y} + e^{-2(y-x)^2 + 0.4y - 0.4x}} \right)$$

$$\left(-4\,(y-1)^3 e^{-(y-1)^4}+(-2y-2)\,e^{-(y+1)^2}\right)^2\Bigg)$$

$$+\frac{1}{\left(e^{-(y-1)^4}+e^{-(y+1)^2}\right)^2}\left(\left(\left(e^{-2(y-x)^2+0.4y-0.4x}\left(-1.\,e^{-(x-1)^4}-1.\,e^{-(x+1)^2}\right.\right.\right.\right.$$

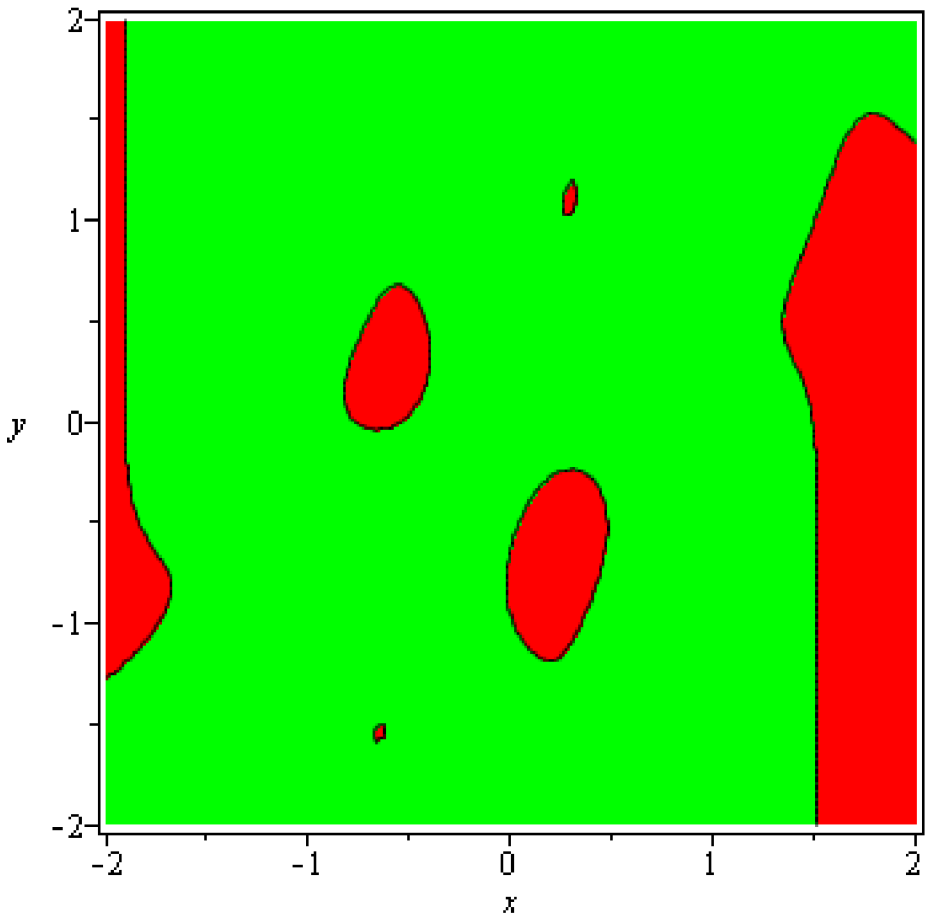
$$\left.\left.\left.\left.+\left(e^{-(y-1)^4}+e^{-(y+1)^2}\right)e^{-2(x-y)^2+0.4x-0.4y}\right)\right)\Big/\left(-1.\right.\right.\right.$$

$$\left.\left.\left.+e^{-2(x-y)^2+0.4x-0.4y}e^{-2(y-x)^2+0.4y-0.4x}\right)\right.$$

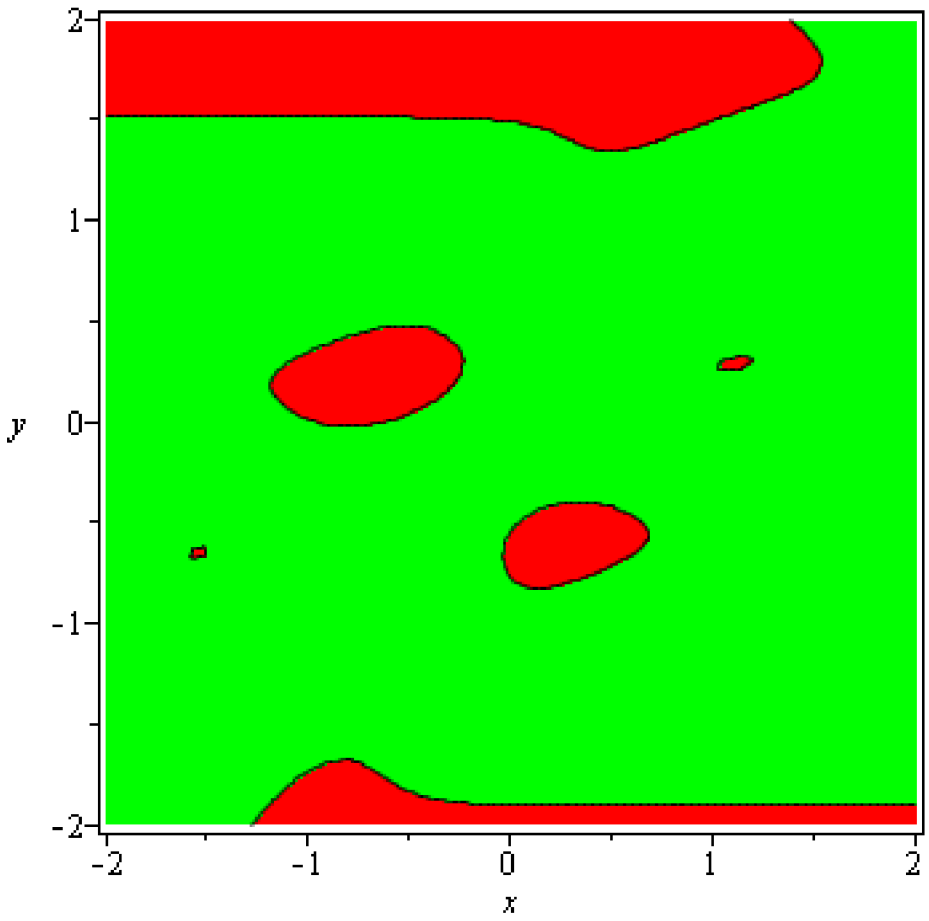
$$-\frac{e^{0.}\left(-\left(e^{-(x-1)^4}+e^{-(x+1)^2}\right)e^{-2(y-x)^2+0.4y-0.4x}+1.\,e^{-(y-1)^4}+1.\,e^{-(y+1)^2}\right)}{-1.+e^{-2(x-y)^2+0.4x-0.4y}e^{-2(y-x)^2+0.4y-0.4x}}\Bigg)\Bigg($$

$$-12\,(y-1)^2 e^{-(y-1)^4}+16\,(y-1)^6 e^{-(y-1)^4}-2\,e^{-(y+1)^2}+(-2y-2)^2 e^{-(y+1)^2}\Bigg)\Bigg)$$

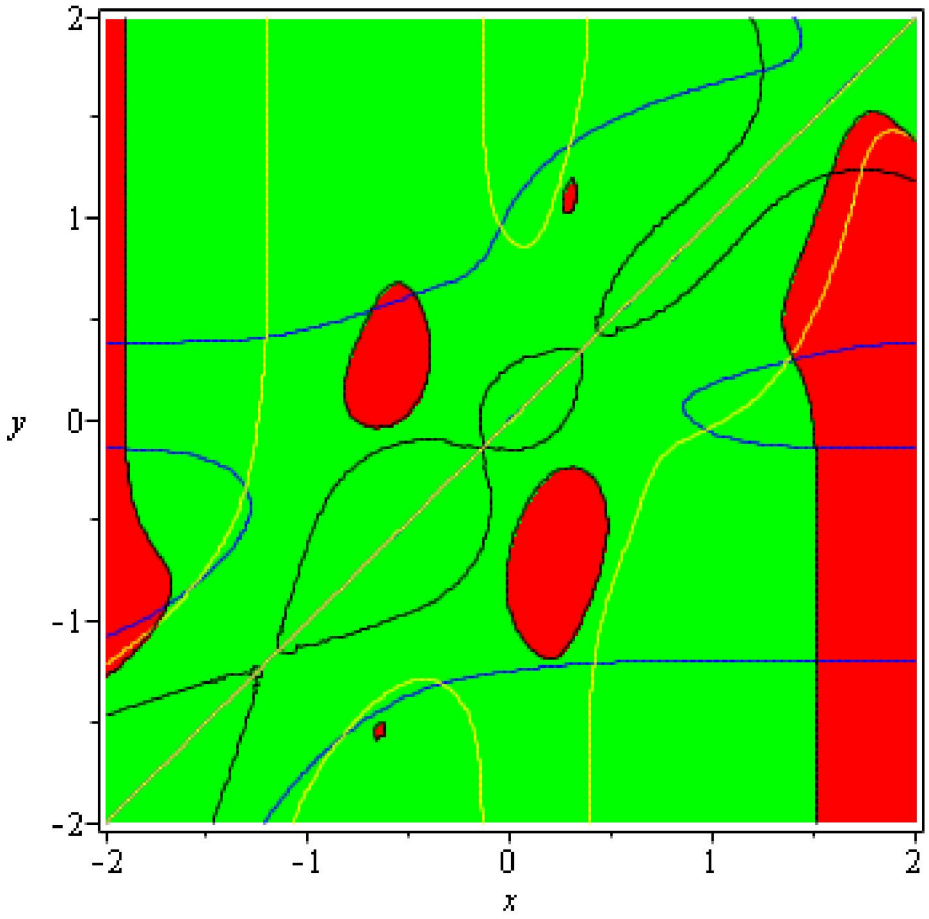
```
O contourplot(piecewise(abs(x-y)<0.01,0,D2s[x](x,y)),x=-2..2,y=-2.
  .2,filled=true,contours=[0],coloring=[red,green],grid=[100,100],
  axes=boxed);
```



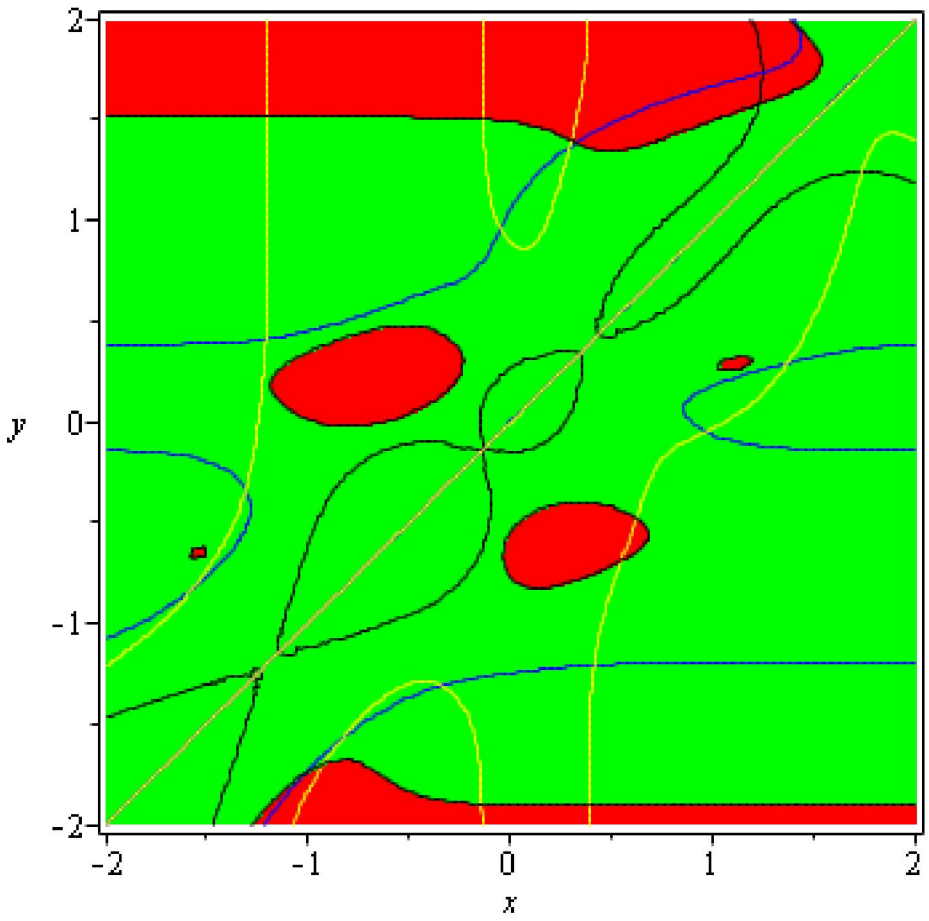

```
> contourplot(piecewise(abs(x-y)<0.01,0,D2s[y](x,y)),x=-2..2,y=-2.
  .2,filled=true,contours=[0],coloring=[red,green],grid=[100,100],
  axes=boxed);
```



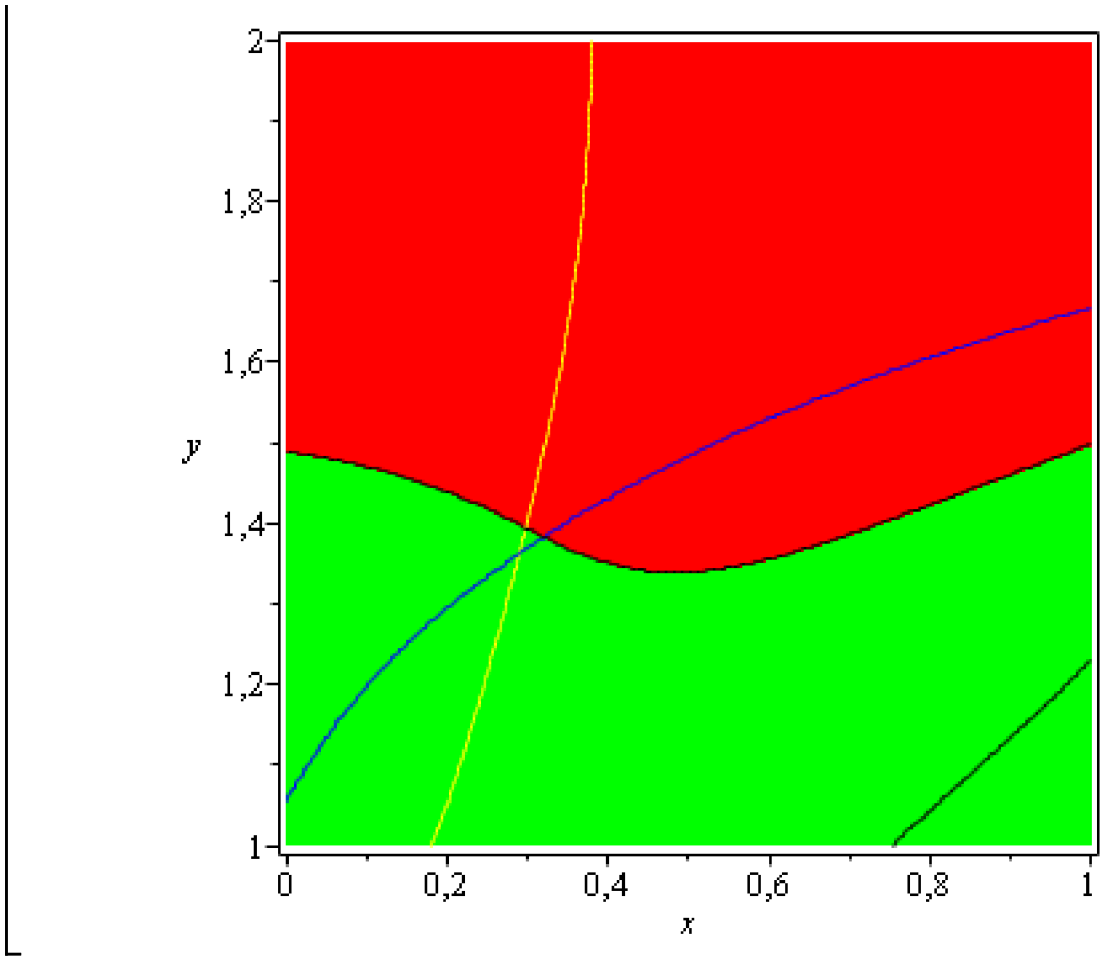

```
> E:=contourplot(piecewise(abs(x-y)<0.01,0,D2s[x](x,y)),x=-2..2,y=
  -2..2,filled=true,contours=[0],coloring=[red,green],grid=[100,
  100],axes=boxed):
> F:=contourplot(piecewise(abs(x-y)<0.01,0,D2s[y](x,y)),x=-2..2,y=
  -2..2,filled=true,contours=[0],coloring=[red,green],grid=[100,
  100],axes=boxed):
> G:=contourplot(s_mono(x,y)*s_mono(y,x),x=-2..2,y=-2..2,contours=
  [0], color=black, grid=[100,100],axes=boxed):
> display({A,B,E,G}); #every dimorphic singularity has D2s[x]>0 ->
  further branching into a trimorphic population
```




```
> display({A,B,F,G}); #every dimorphic singularity has D2s[y]>0 ->
  further branching into a trimorphic population
```




```
> Azoom:=contourplot(piecewise(abs(x-y)<0.01,0,Ds[x](x,y)),x=0..1,
  y=1..2,contours=[0],color=yellow,grid=[100,100],axes=boxed):
> Bzoom:=contourplot(piecewise(abs(x-y)<0.01,0,Ds[y](x,y)),x=0..1,
  y=1..2,contours=[0],color=blue,grid=[100,100],axes=boxed):
> Fzoom:=contourplot(piecewise(abs(x-y)<0.01,0,D2s[y](x,y)),x=0.
  .1,y=1..2,filled=true,contours=[0],coloring=[red,green],grid=
  [100,100],axes=boxed):
> Gzoom:=contourplot(s_mono(x,y)*s_mono(y,x),x=0..1,y=1..2,
  contours=[0], color=black, grid=[100,100],axes=boxed):
> display({Azoom,Bzoom,Fzoom,Gzoom});
```


2
1,8
1,6
1,4
1,2
1
y
0
0,2
0,4
0,6
0,8
1
x