```
> s:=(x,y)->r(y)*(1-(a(y,x)*K(x))/K(y));  #invasion fitness
```

$$s := (x, y) \to r(y) \left(1 - \frac{a(y, x)\, K(x)}{K(y)}\right) \quad (1)$$

```
> K:=x->exp(-x^2);
```

$$K := x \to e^{-x^2} \quad (2)$$

```
> a:=(x,y)->exp(-alpha*(x-y)^2);
```

$$a := (x, y) \to e^{-\alpha (x - y)^2} \quad (3)$$

```
> r:=x->1;
```

$$r := x \to 1 \quad (4)$$

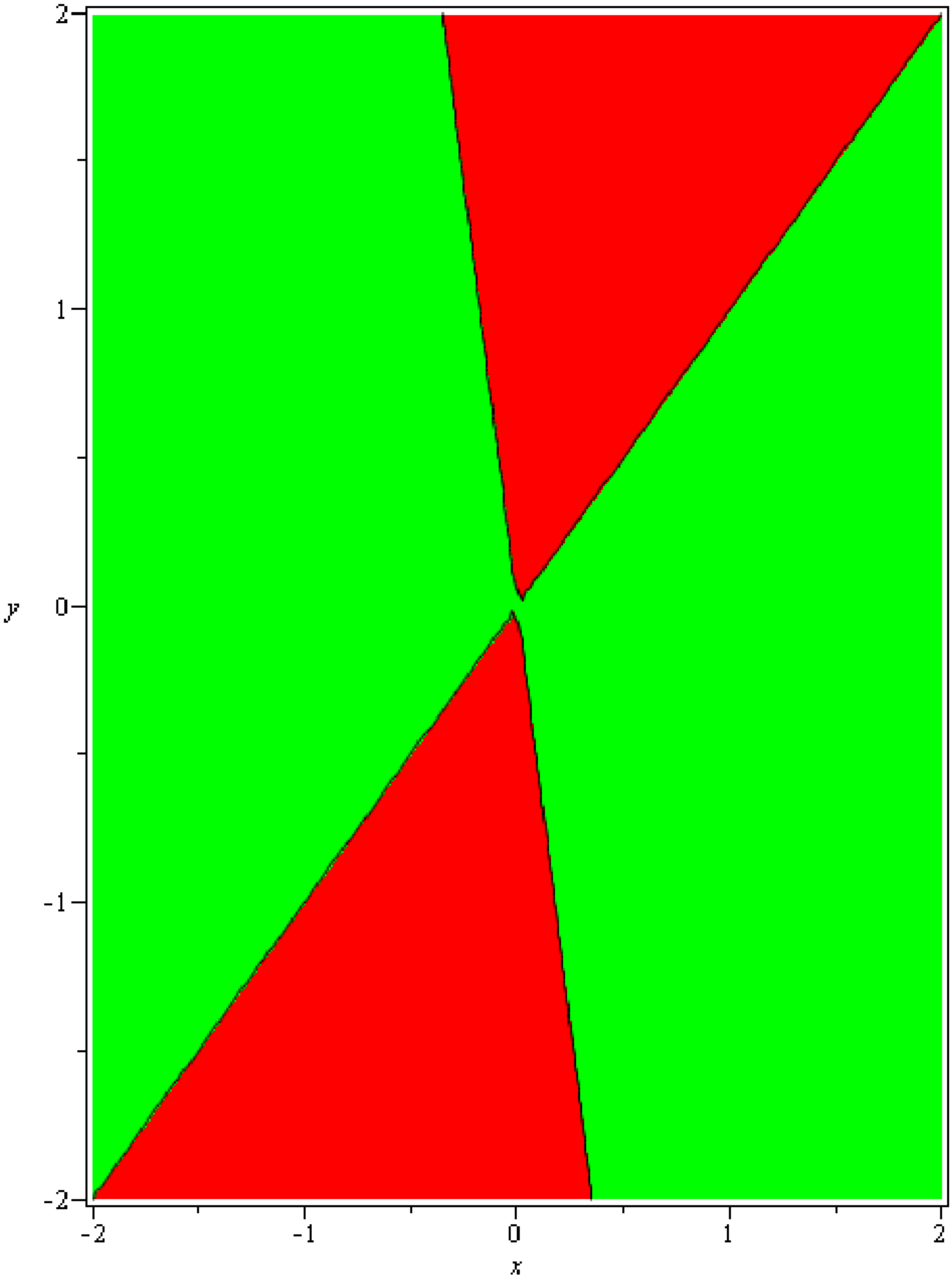
```
> s(x,x); #the fitness of the resident strategy in the res.pop.
must be 0!! it's always good to check this
```

$$0 \quad (5)$$

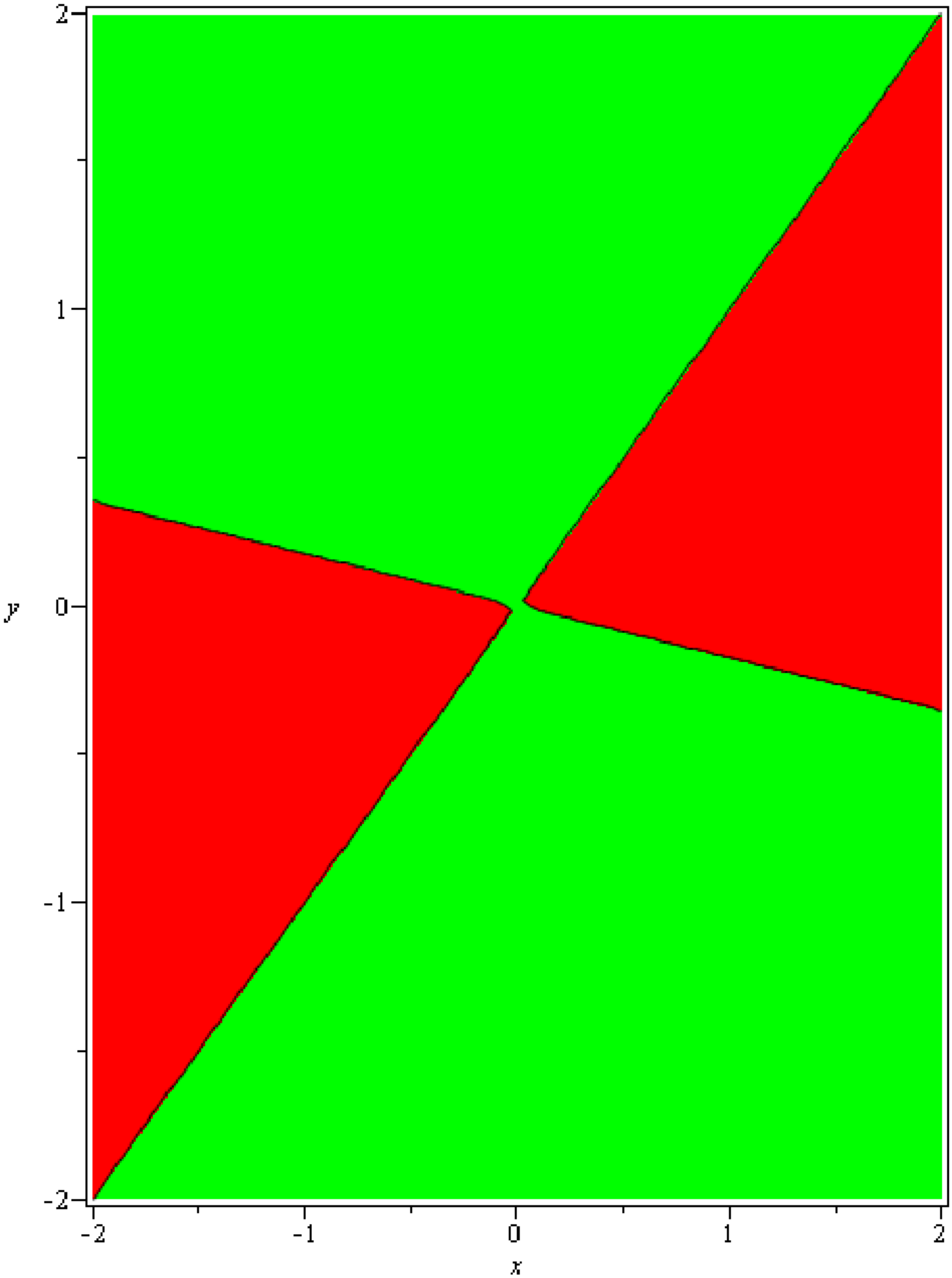
```
> alpha:=0.7;
```

$$\alpha := 0.7 \quad (6)$$

```
> with(plots):
> contourplot(s(x,y),x=-2..2,y=-2..2,contours=[0],filled=true,
coloring=[red,green],grid=[100,100],axes=boxed); #pairwise
invadability plot (PIP)
```

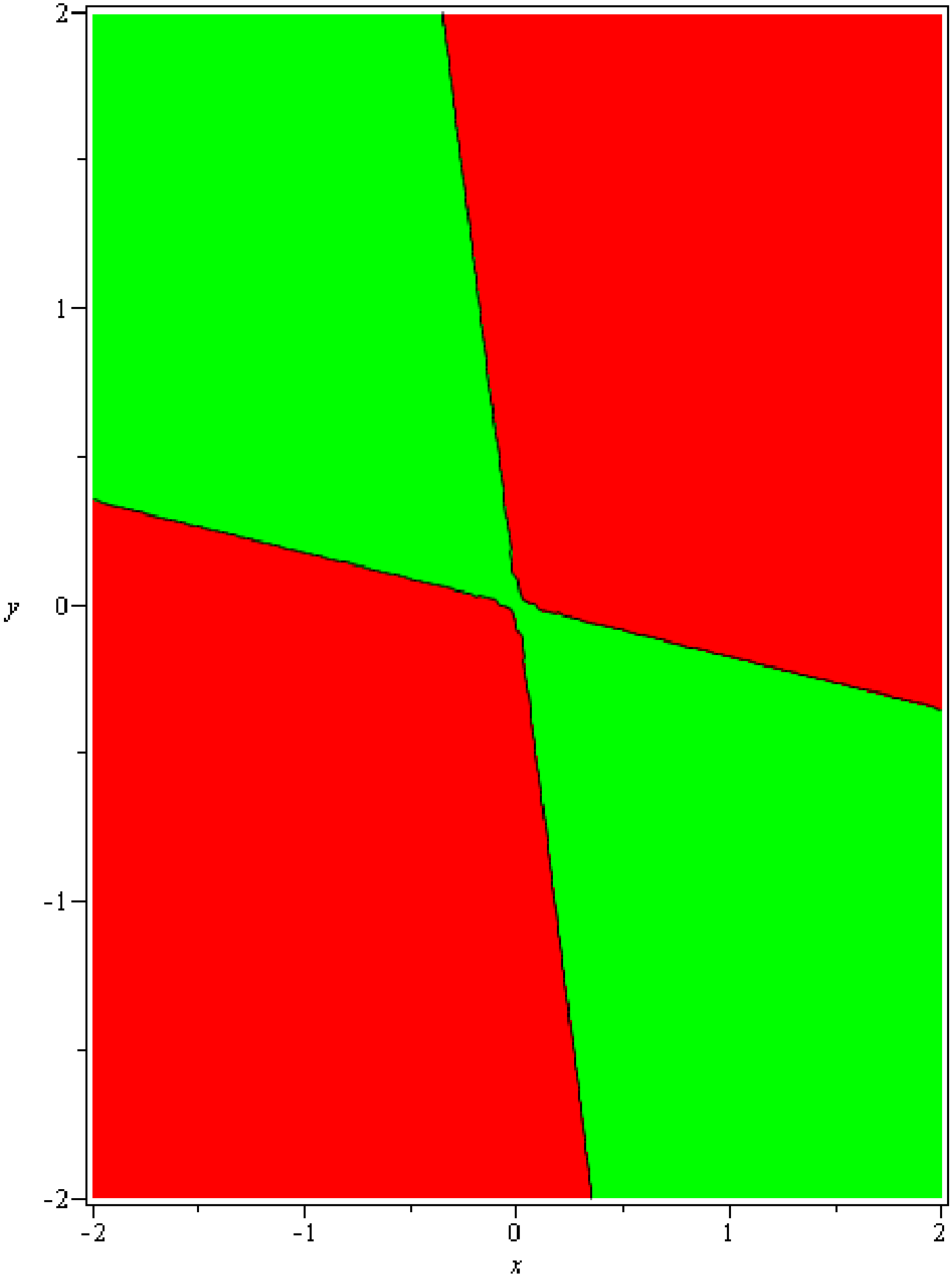



```
> contourplot(s(y,x),x=-2..2,y=-2..2,contours=[0],filled=true,
  coloring=[red,green],grid=[100,100],axes=boxed); #mirror image
  (along x=y) of the PIP
```




```
contourplot(s(x,y)*s(y,x),x=-2..2,y=-2..2,contours=[0],filled=
true,coloring=[red,green],grid=[100,100],axes=boxed); #mutual
invadability plot (MIP), be careful with this sort of plotting,
```

if two minus regions overlap, this would give mutual coexistence there!

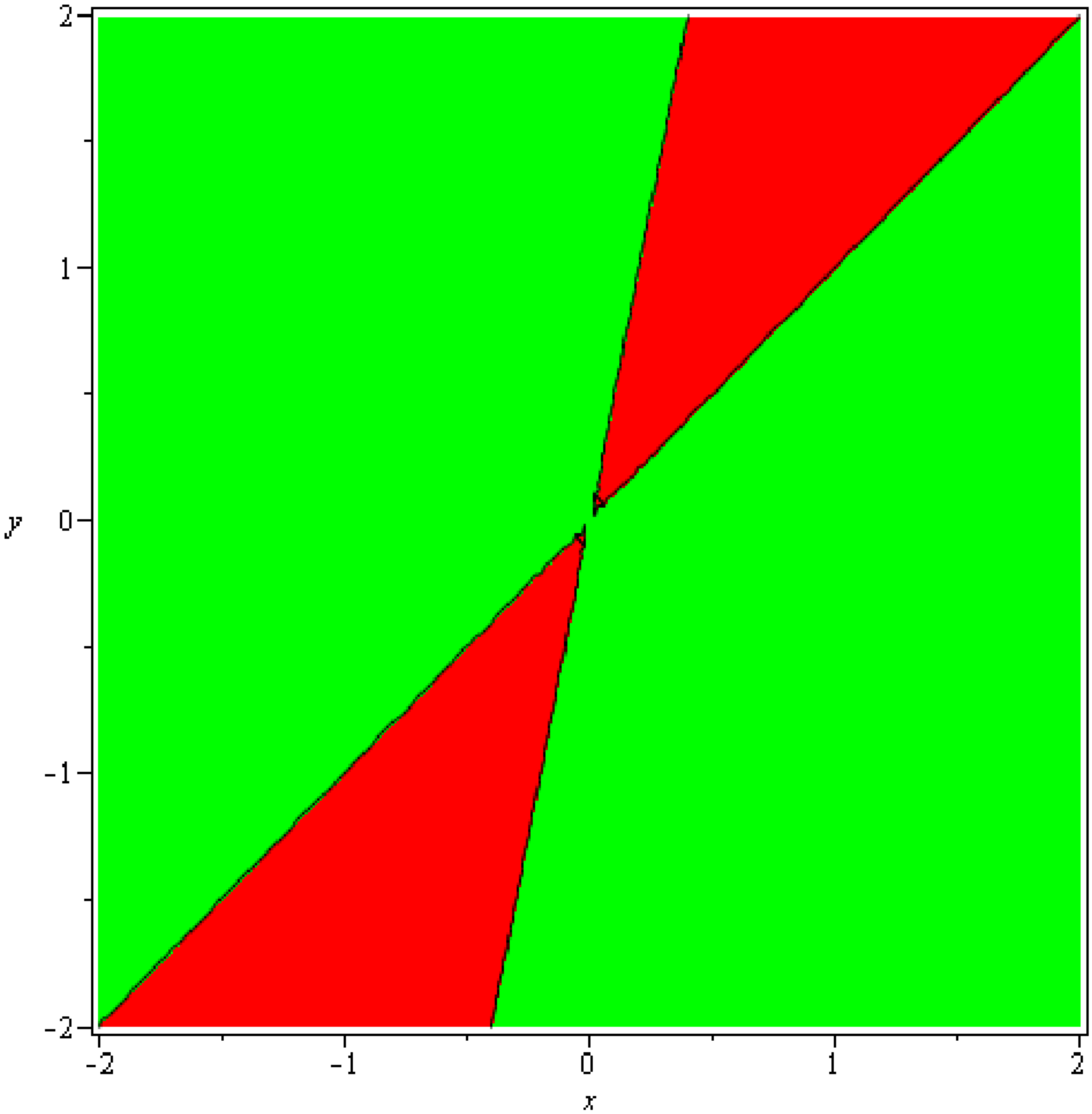



```
alpha:=1.5;
```

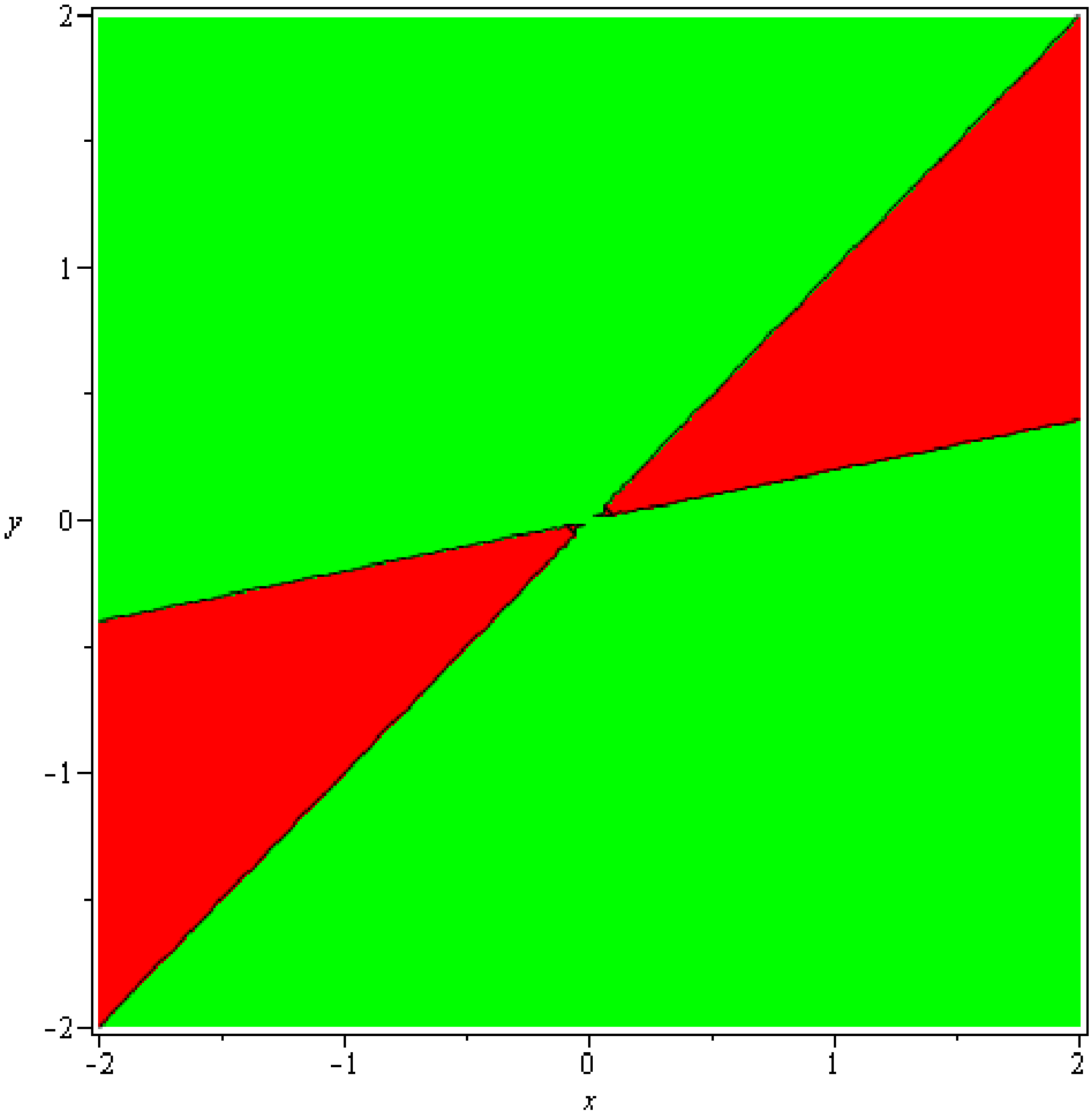
(7)

$$a := 1.5 \qquad (7)$$

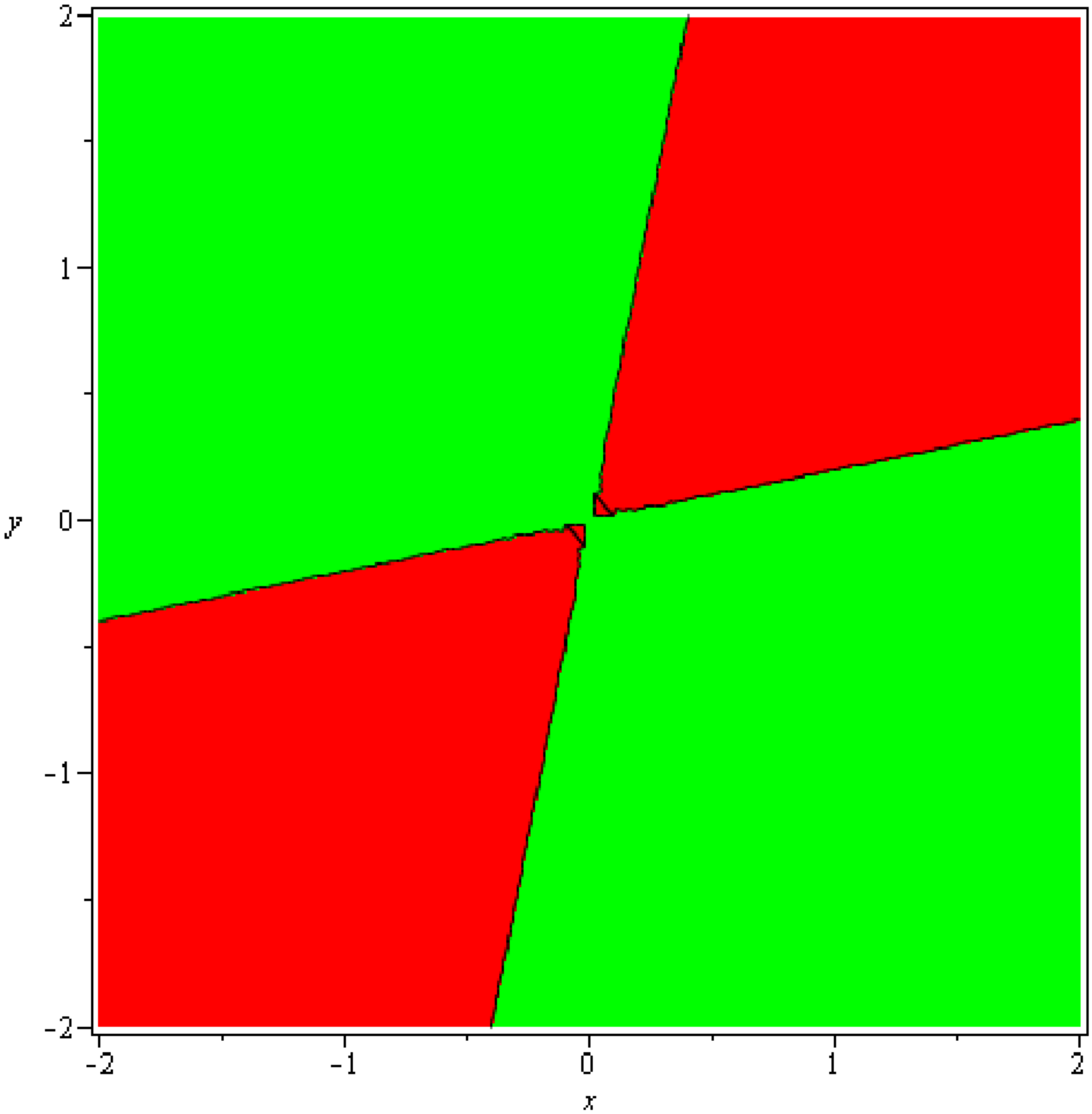
```
contourplot(s(x,y),x=-2..2,y=-2..2,contours=[0],filled=true,
coloring=[red,green],grid=[100,100],axes=boxed); #pairwise
invadability plot (PIP)
```




```
contourplot(s(y,x),x=-2..2,y=-2..2,contours=[0],filled=true,
coloring=[red,green],grid=[100,100],axes=boxed);
```




```
contourplot(s(x,y)*s(y,x),x=-2..2,y=-2..2,contours=[0],filled=
true,coloring=[red,green],grid=[100,100],axes=boxed); #MIP for
alpha=1.5
```




```
O restart;
```

```
O k:=2;
```

$$k := 2 \tag{8}$$

```
O for i from 1 to k do
  1-(( sum(a(x[i],x[j])*n[j], j=1..k) )/K(x[i]))=0
  end do; #equilibrium conditions for n[1] and n[2]
```

$$1 \text{ K } \frac{a(x_1, x_1)\, n_1 \text{ C } a(x_1, x_2)\, n_2}{K(x_1)} = 0$$

$$1 \text{ K } \frac{a(x_2, x_1)\, n_1 \text{ C } a(x_2, x_2)\, n_2}{K(x_2)} = 0 \tag{9}$$

```
> solve([1-(a(x[1],x[1])*n[1]+a(x[1],x[2])*n[2])/K(x[1]) = 0, 1-(a(x[2],x[1])*n[1]+a(x[2],x[2])*n[2])/K(x[2]) = 0],[n[1],n[2]]);
  #two equations, two unknowns: we can solve for n[1] and n[2]
```

$$\left[\left[n_1 = \frac{-a(x_2, x_2)\,K(x_1) + K(x_2)\,a(x_1, x_2)}{-a(x_1, x_1)\,a(x_2, x_2) + a(x_1, x_2)\,a(x_2, x_1)},\ n_2 = -\frac{-K(x_1)\,a(x_2, x_1) + a(x_1, x_1)\,K(x_2)}{-a(x_1, x_1)\,a(x_2, x_2) + a(x_1, x_2)\,a(x_2, x_1)}\right]\right] \quad (10)$$

```
> subs(x[1]=x,x[2]=y,%); #substitutes x and y for x[1] and x[2] in the previous equations
```

$$\left[\left[n_1 = \frac{-a(y, y)\,K(x) + K(y)\,a(x, y)}{-a(x, x)\,a(y, y) + a(x, y)\,a(y, x)},\ n_2 = -\frac{-K(x)\,a(y, x) + a(x, x)\,K(y)}{-a(x, x)\,a(y, y) + a(x, y)\,a(y, x)}\right]\right] \quad (11)$$

```
> n[1]:=(x,y)->(-a(y,y)*K(x)+K(y)*a(x,y))/(-a(x,x)*a(y,y)+a(x,y)*a(y,x)); #defining n[1] as a function of x and y
```

$$n_1 := (x, y) \mapsto \frac{-a(y, y)\,K(x) + K(y)\,a(x, y)}{-a(x, x)\,a(y, y) + a(x, y)\,a(y, x)} \quad (12)$$

```
> n[2]:=(x,y)->-1/(-a(x,x)*a(y,y)+a(x,y)*a(y,x))*(-K(x)*a(y,x)+a(x,x)*K(y));
```

$$n_2 := (x, y) \mapsto -\frac{-K(x)\,a(y, x) + a(x, x)\,K(y)}{-a(x, x)\,a(y, y) + a(x, y)\,a(y, x)} \quad (13)$$

```
> K:=x->exp(-x^2);
```

$$K := x \mapsto e^{-x^2} \quad (14)$$

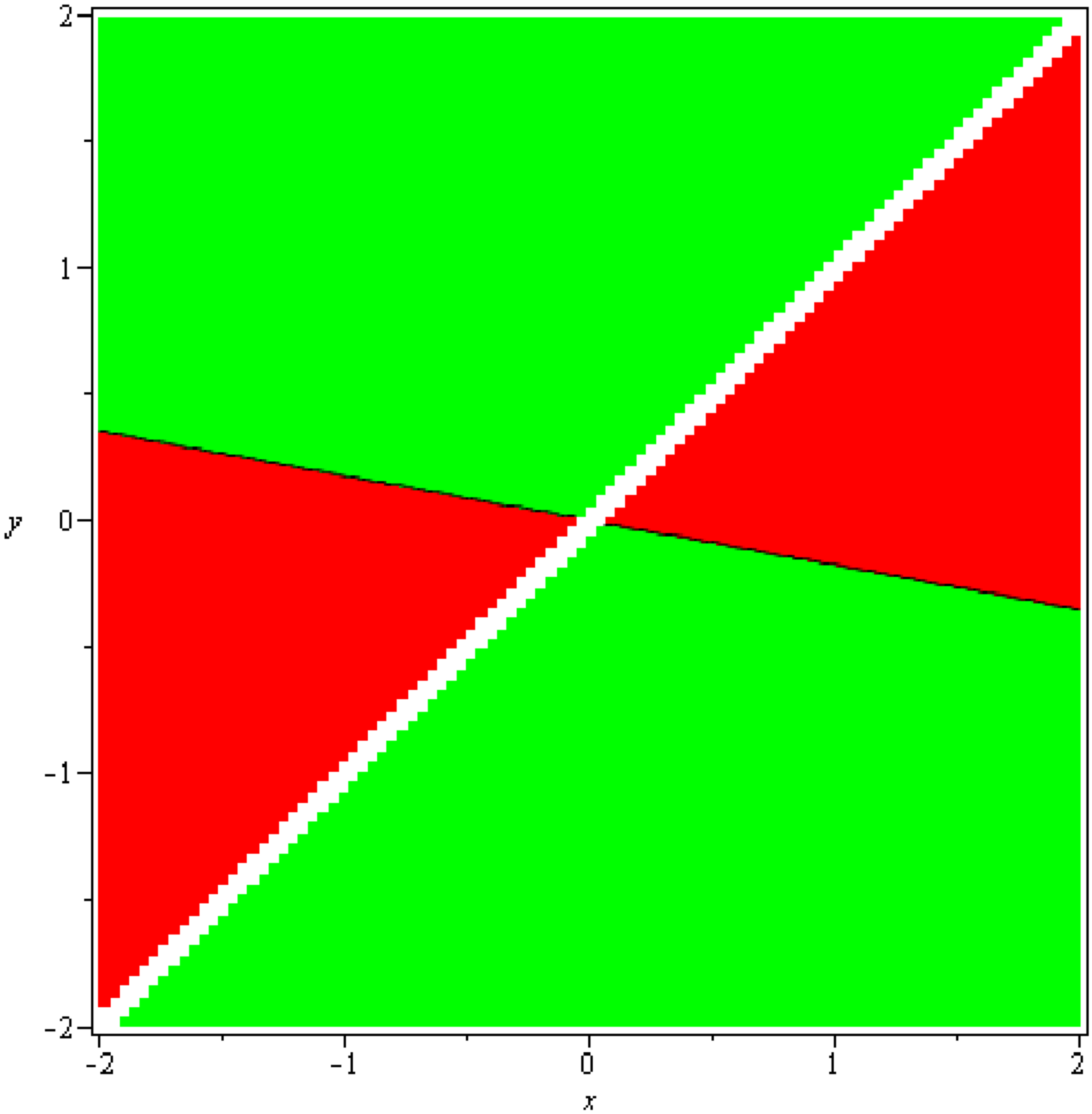
```
> a:=(x,y)->exp(-alpha*(x-y)^2);
```

$$a := (x, y) \mapsto e^{-\alpha (x - y)^2} \quad (15)$$

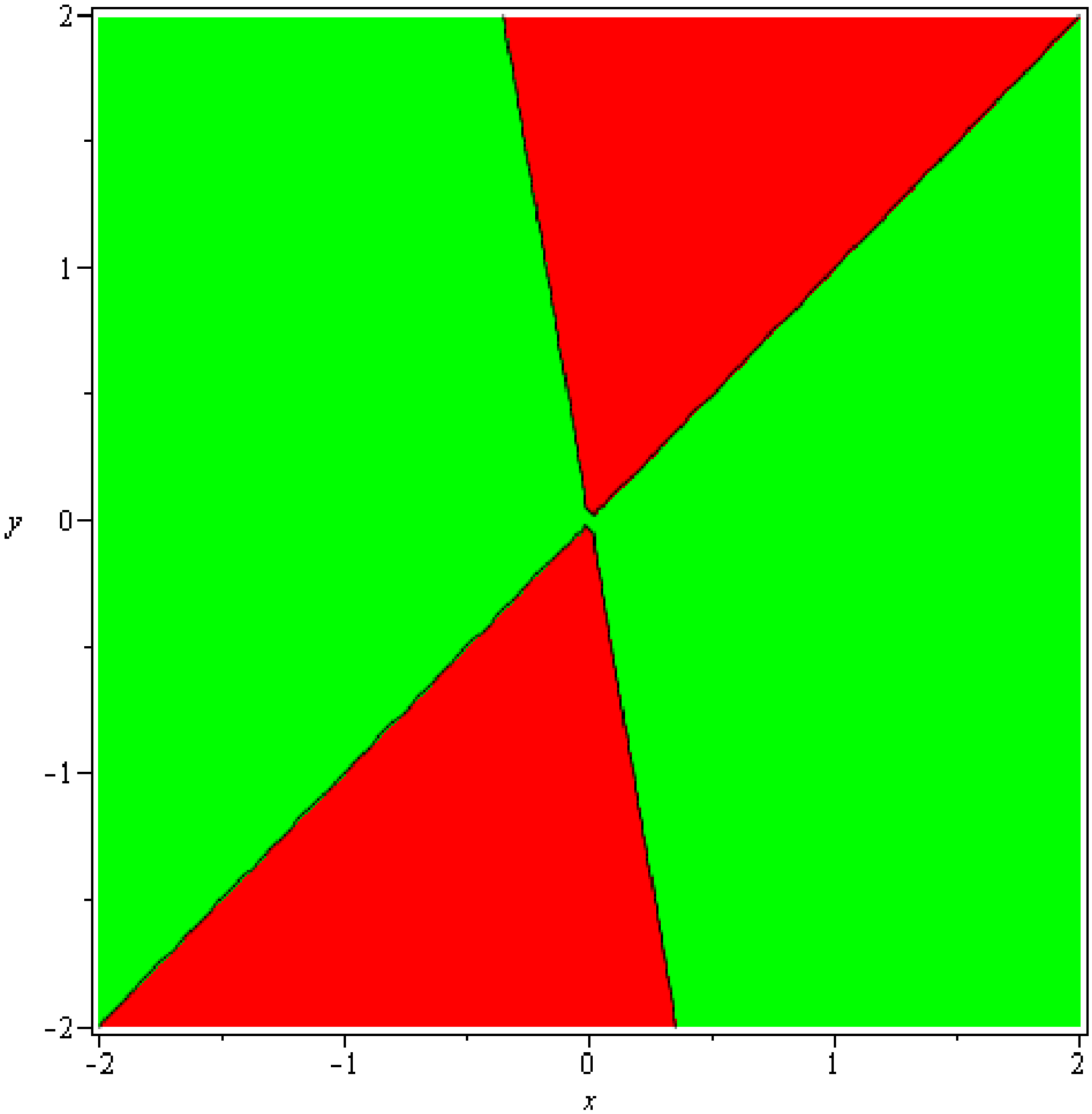
```
> alpha:=0.7;
```

$$\alpha := 0.7 \quad (16)$$

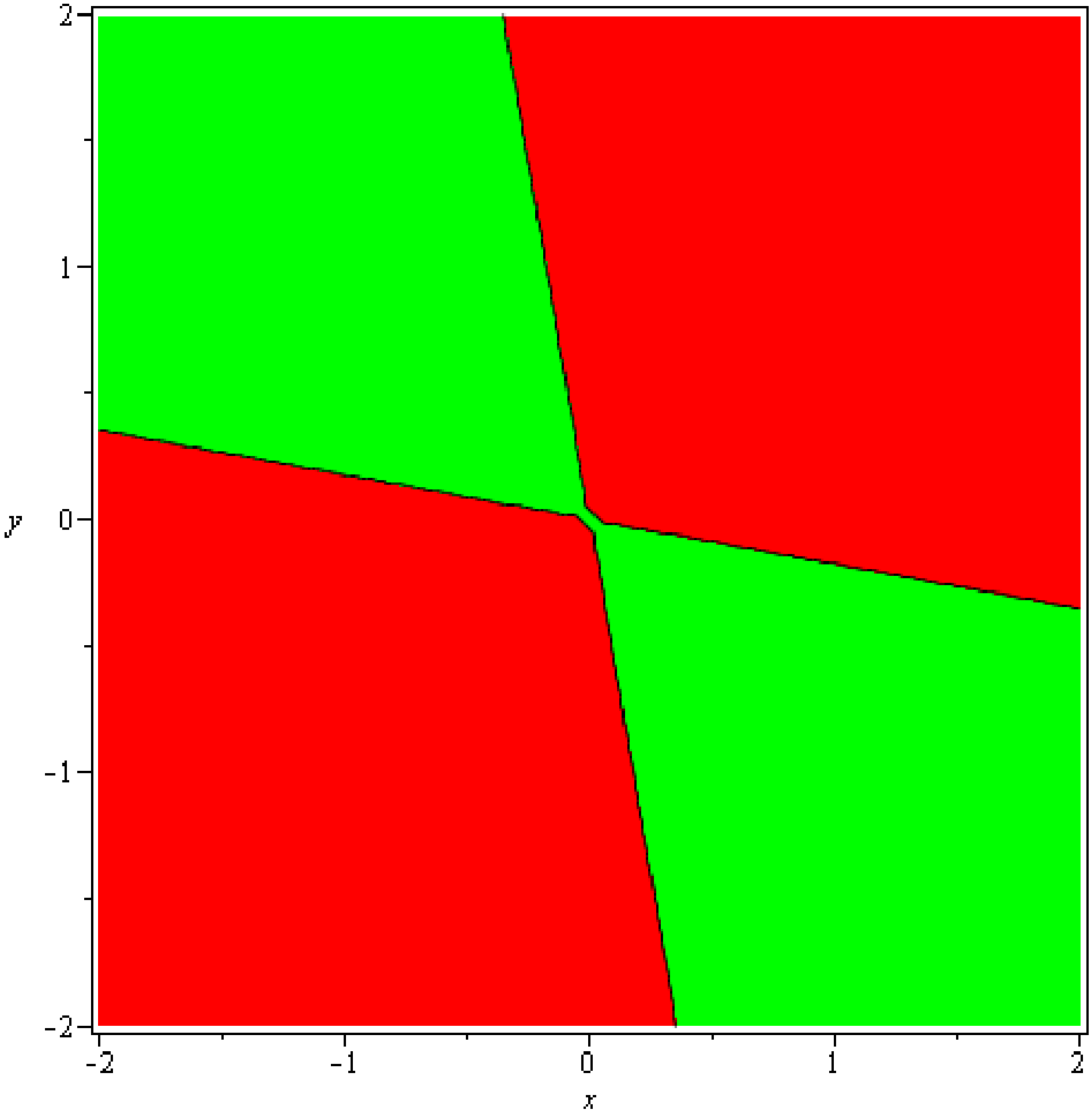
```
> with(plots):
> contourplot(n[1](x,y),x=-2..2,y=-2..2,contours=[0],filled=true,coloring=[red,green],grid=[100,100],axes=boxed); #see the following plot for how to correct the problem around x=y here
```




```
contourplot(piecewise(abs(x-y)<0.01,0,n[2](x,y)),x=-2..2,y=-2.
.2,contours=[0],filled=true,coloring=[red,green],grid=[100,100],
axes=boxed); #this plots a piecewise function, which has the
value 0 close to the diagonal and n[2](x,y) elsewhere
```



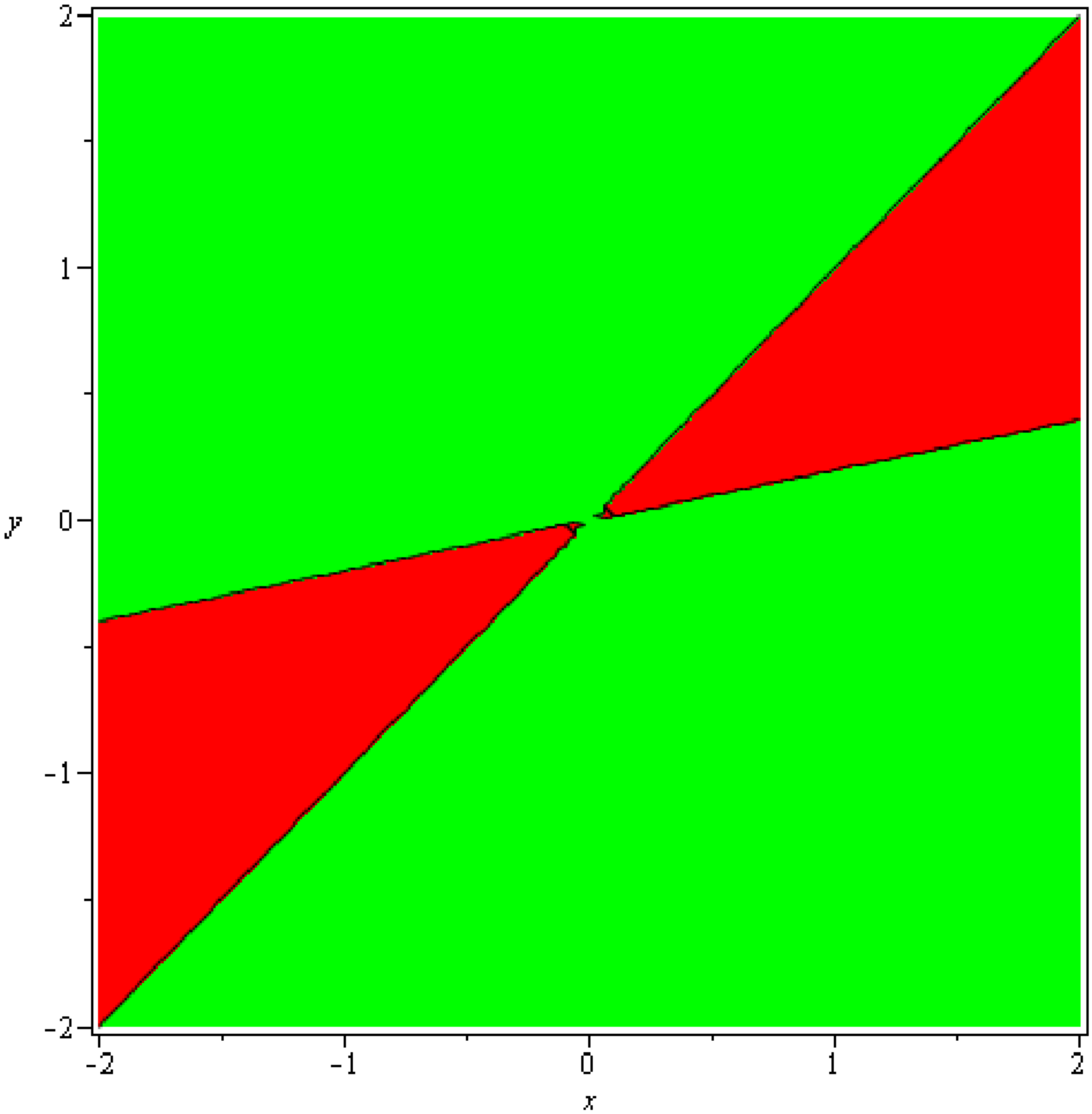

```
contourplot(piecewise(abs(x-y)<0.01,0,n[1](x,y)*n[2](x,y)),x=-2..2,y=-2..2,contours=[0],filled=true,coloring=[red,green],grid=[100,100],axes=boxed);
```



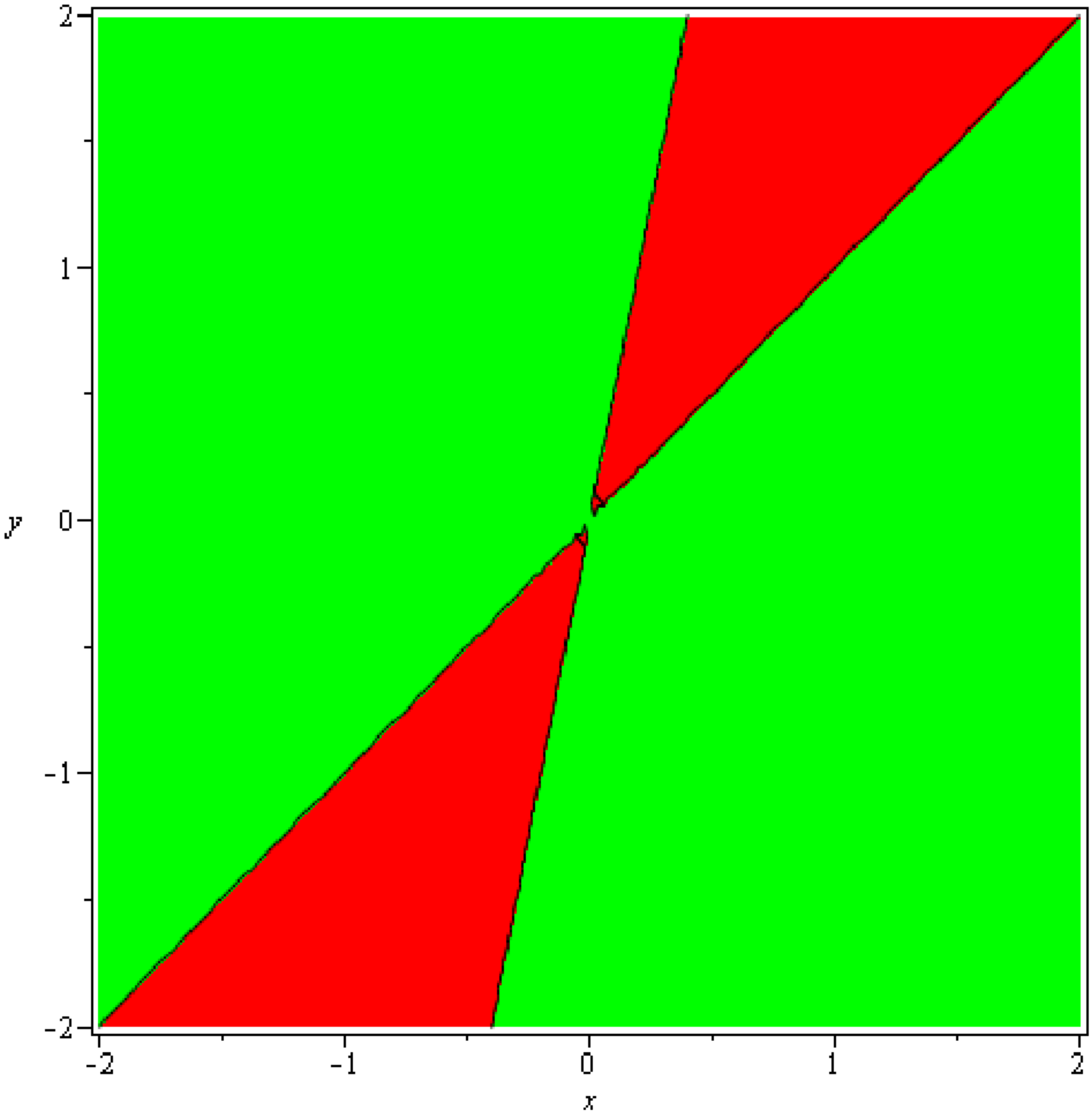

```
> alpha:=1.5;
```

$$a := 1.5 \quad (17)$$

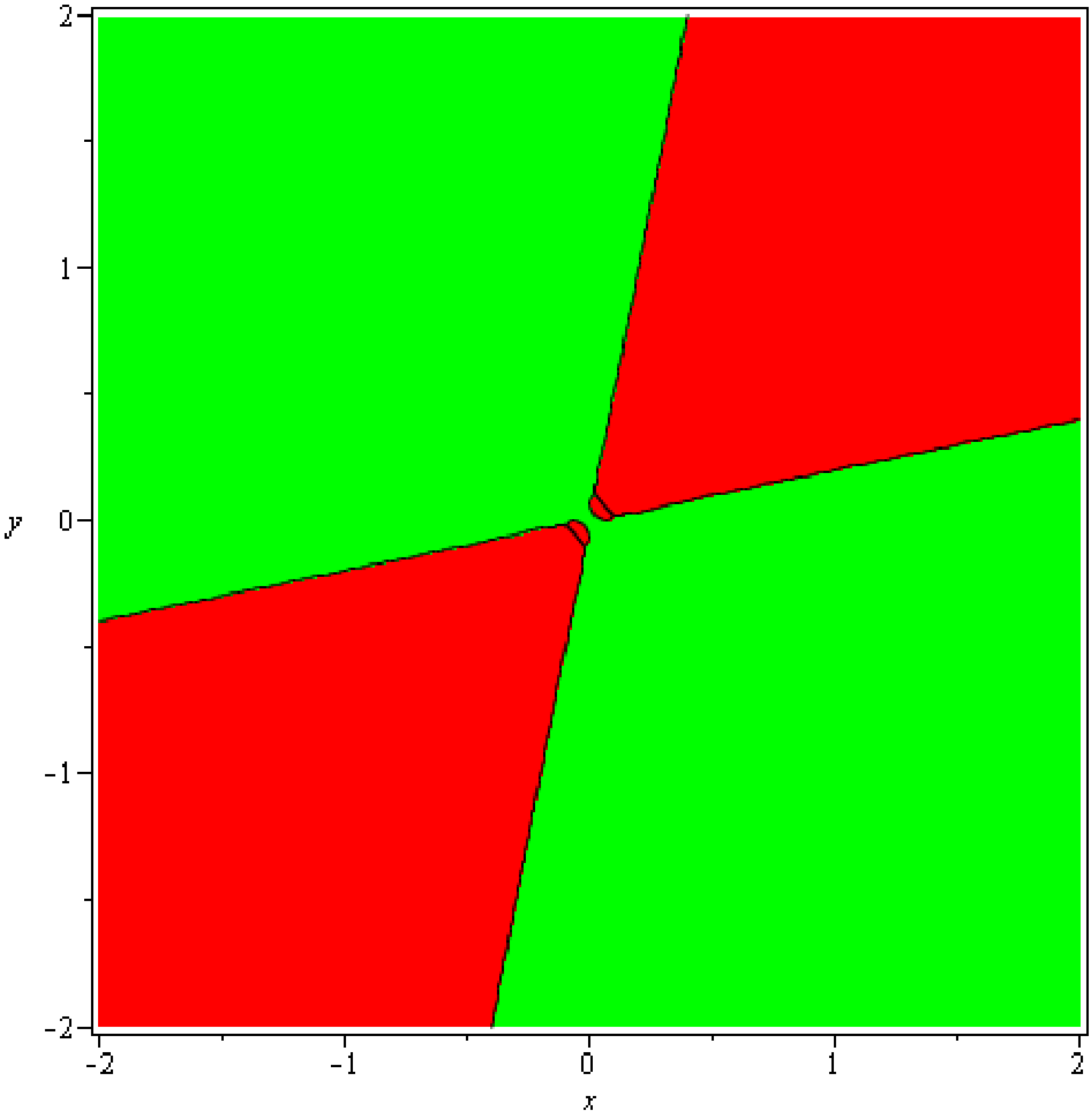
```
> contourplot(piecewise(abs(x-y)<0.01,0,n[1](x,y)),x=-2..2,y=-2..2,contours=[0],filled=true,coloring=[red,green],grid=[100,100],axes=boxed);
```




```
contourplot(piecewise(abs(x-y)<0.01,0,n[2](x,y)),x=-2..2,y=-2.
.2,contours=[0],filled=true,coloring=[red,green],grid=[100,100],
axes=boxed);
```




```
> contourplot(piecewise(abs(x-y)<0.01,0,n[1](x,y)*n[2](x,y)),x=-2..2,y=-2..2,contours=[0],filled=true,coloring=[red,green],grid=[100,100],axes=boxed);
```



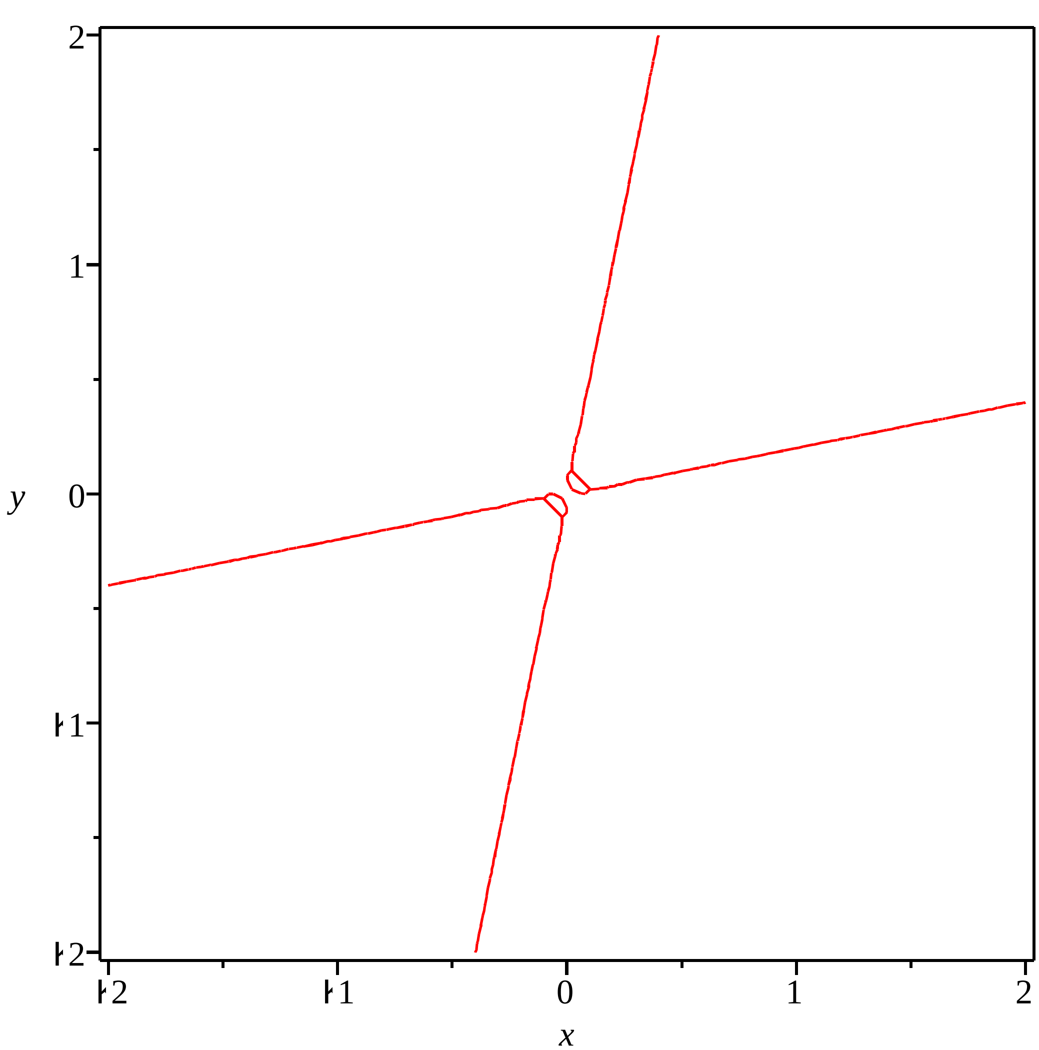
O s d (x, y) / r(y) * (1 K (a(y, x) * K(x)) / K(y));

$$s := (x, y) \rightarrow r(y) \left(1 - \frac{K(x)\, a(y, x)}{K(y)}\right) \quad \textbf{(18)}$$

O Q d contourplot(s(x, y) * s(y, x), x = K2 ..2, y = K2 ..2, contours = [0], grid = [100, 100], axes = boxed) : #MIP for alpha=1.5

O R d contourplot(piecewise(abs(x K y) ! 0.01, 0, n[1](x, y) * n[2](x, y)), x = K2 ..2, y = K2 ..2, contours = [0], grid = [100, 100], axes = boxed) :

O display({Q, R}); #the MIP and the area where n[1] and n[2] are both positive coincide




```
restart;
```

```
s:=(x,y,z)->r(z)*( 1-((a(z,x)*n[1](x,y)+a(z,y)*n[2](x,y))/K(z))
); #fitness of a rare mutant z in a resident population of x and
y
```

$$s := (x, y, z) \to r(z) \left(1 - \frac{a(z, x)\, n_1(x, y) + a(z, y)\, n_2(x, y)}{K(z)}\right) \tag{19}$$

```
K:=x->exp(-x^2);
```

$$K := x \to e^{-x^2} \tag{20}$$

```
a:=(x,y)->exp(-alpha*(x-y)^2);
```

$$a := (x, y) \to e^{-\alpha (x - y)^2} \tag{21}$$

```
r:=x->1;
```

$$r := x \to 1 \tag{22}$$

```
n[1]:=(x,y)->(-a(y,y)*K(x)+K(y)*a(x,y))/(-a(x,x)*a(y,y)+a(x,y)*a
(y,x));
```

(23)

$$n_1 := (x, y) \to \frac{-a(y, y)\, K(x) + K(y)\, a(x, y)}{-a(x, x)\, a(y, y) + a(x, y)\, a(y, x)} \tag{23}$$

```
> n[2]:=(x,y)->-1/(-a(x,x)*a(y,y)+a(x,y)*a(y,x))*(-K(x)*a(y,x)+a
(x,x)*K(y));
```

$$n_2 := (x, y) \to -\frac{-K(x)\, a(y, x) + a(x, x)\, K(y)}{-a(x, x)\, a(y, y) + a(x, y)\, a(y, x)} \tag{24}$$

```
> diff(s(x,y,z),z);
```

$$-\frac{1}{e^{-z^2}}\left(-\frac{2\,\alpha\,(z-x)\,e^{-\alpha\,(z-x)^2}\left(-e^{-x^2}+e^{-y^2}\,e^{-\alpha\,(x-y)^2}\right)}{-1+e^{-\alpha\,(x-y)^2}\,e^{-\alpha\,(y-x)^2}} + \frac{2\,\alpha\,(z-y)\,e^{-\alpha\,(z-y)^2}\left(-e^{-x^2}\,e^{-\alpha\,(y-x)^2}+e^{-y^2}\right)}{-1+e^{-\alpha\,(x-y)^2}\,e^{-\alpha\,(y-x)^2}}\right) - \frac{2\left(\frac{e^{-\alpha\,(z-x)^2}\left(-e^{-x^2}+e^{-y^2}\,e^{-\alpha\,(x-y)^2}\right)}{-1+e^{-\alpha\,(x-y)^2}\,e^{-\alpha\,(y-x)^2}} - \frac{e^{-\alpha\,(z-y)^2}\left(-e^{-x^2}\,e^{-\alpha\,(y-x)^2}+e^{-y^2}\right)}{-1+e^{-\alpha\,(x-y)^2}\,e^{-\alpha\,(y-x)^2}}\right) z}{e^{-z^2}} \tag{25}$$

```
> subs(z=x,%);
```

$$-\frac{2\,\alpha\,(x-y)\,e^{-\alpha\,(x-y)^2}\left(-e^{-x^2}\,e^{-\alpha\,(y-x)^2}+e^{-y^2}\right)}{\left(-1+e^{-\alpha\,(x-y)^2}\,e^{-\alpha\,(y-x)^2}\right)e^{-x^2}} - \frac{2\left(\frac{e^{0}\left(-e^{-x^2}+e^{-y^2}\,e^{-\alpha\,(x-y)^2}\right)}{-1+e^{-\alpha\,(x-y)^2}\,e^{-\alpha\,(y-x)^2}} - \frac{e^{-\alpha\,(x-y)^2}\left(-e^{-x^2}\,e^{-\alpha\,(y-x)^2}+e^{-y^2}\right)}{-1+e^{-\alpha\,(x-y)^2}\,e^{-\alpha\,(y-x)^2}}\right) x}{e^{-x^2}} \tag{26}$$

```
> Ds[x]:=(x,y)->-2*alpha*(x-y)*exp(-alpha*(x-y)^2)/(-1+exp(-alpha*
(x-y)^2)*exp(-alpha*(y-x)^2))*(-exp(-x^2)*exp(-alpha*(y-x)^2)+
exp(-y^2))/exp(-x^2)-2*(exp(0)*(-exp(-x^2)+exp(-y^2)*exp(-alpha*
(x-y)^2))/(-1+exp(-alpha*(x-y)^2)*exp(-alpha*(y-x)^2))-exp(-
alpha*(x-y)^2)/(-1+exp(-alpha*(x-y)^2)*exp(-alpha*(y-x)^2))*(-
exp(-x^2)*exp(-alpha*(y-x)^2)+exp(-y^2)))/exp(-x^2)*x; #the
selection gradient in x-direction
```

$$Ds_x := (x, y) \to -\frac{2\,\alpha\,(x-y)\,e^{-\alpha\,(x-y)^2}\left(-e^{-x^2}\,e^{-\alpha\,(y-x)^2}+e^{-y^2}\right)}{\left(-1+e^{-\alpha\,(x-y)^2}\,e^{-\alpha\,(y-x)^2}\right)e^{-x^2}} - \frac{2\left(\frac{e^{0}\left(-e^{-x^2}+e^{-y^2}\,e^{-\alpha\,(x-y)^2}\right)}{-1+e^{-\alpha\,(x-y)^2}\,e^{-\alpha\,(y-x)^2}} - \frac{e^{-\alpha\,(x-y)^2}\left(-e^{-x^2}\,e^{-\alpha\,(y-x)^2}+e^{-y^2}\right)}{-1+e^{-\alpha\,(x-y)^2}\,e^{-\alpha\,(y-x)^2}}\right) x}{e^{-x^2}} \tag{27}$$

```
> subs(z=y,-(-2*alpha*(z-x)*exp(-alpha*(z-x)^2)*(-exp(-x^2)+exp(-
y^2)*exp(-alpha*(x-y)^2))/(-1+exp(-alpha*(x-y)^2)*exp(-alpha*(y-
```

```
x)^2))+2*alpha*(z-y)*exp(-alpha*(z-y)^2)/(-1+exp(-alpha*(x-y)^2)
*exp(-alpha*(y-x)^2))*(-exp(-x^2)*exp(-alpha*(y-x)^2)+exp(-y^2))
)/exp(-z^2)-2*(exp(-alpha*(z-x)^2)*(-exp(-x^2)+exp(-y^2)*exp(-
alpha*(x-y)^2))/(-1+exp(-alpha*(x-y)^2)*exp(-alpha*(y-x)^2))-exp
(-alpha*(z-y)^2)/(-1+exp(-alpha*(x-y)^2)*exp(-alpha*(y-x)^2))*(-
exp(-x^2)*exp(-alpha*(y-x)^2)+exp(-y^2)))/exp(-z^2)*z;
```

$$\frac{2\,a\,(y\,K\,x)\,e^{K a\,(y\,K\,x)^2}\left(K e^{K x^2}\,C\,e^{K y^2}\,e^{K a\,(x\,K\,y)^2}\right)}{\left(K 1\,C\,e^{K a\,(x\,K\,y)^2}\,e^{K a\,(y\,K\,x)^2}\right)e^{K y^2}} \\ K\,\frac{2\left(\frac{e^{K a\,(y\,K\,x)^2}\left(K e^{K x^2}\,C\,e^{K y^2}\,e^{K a\,(x\,K\,y)^2}\right)}{K 1\,C\,e^{K a\,(x\,K\,y)^2}\,e^{K a\,(y\,K\,x)^2}}\,K\,\frac{e^{0}\left(K e^{K x^2}\,e^{K a\,(y\,K\,x)^2}\,C\,e^{K y^2}\right)}{K 1\,C\,e^{K a\,(x\,K\,y)^2}\,e^{K a\,(y\,K\,x)^2}}\right)y}{e^{K y^2}} \tag{28}$$

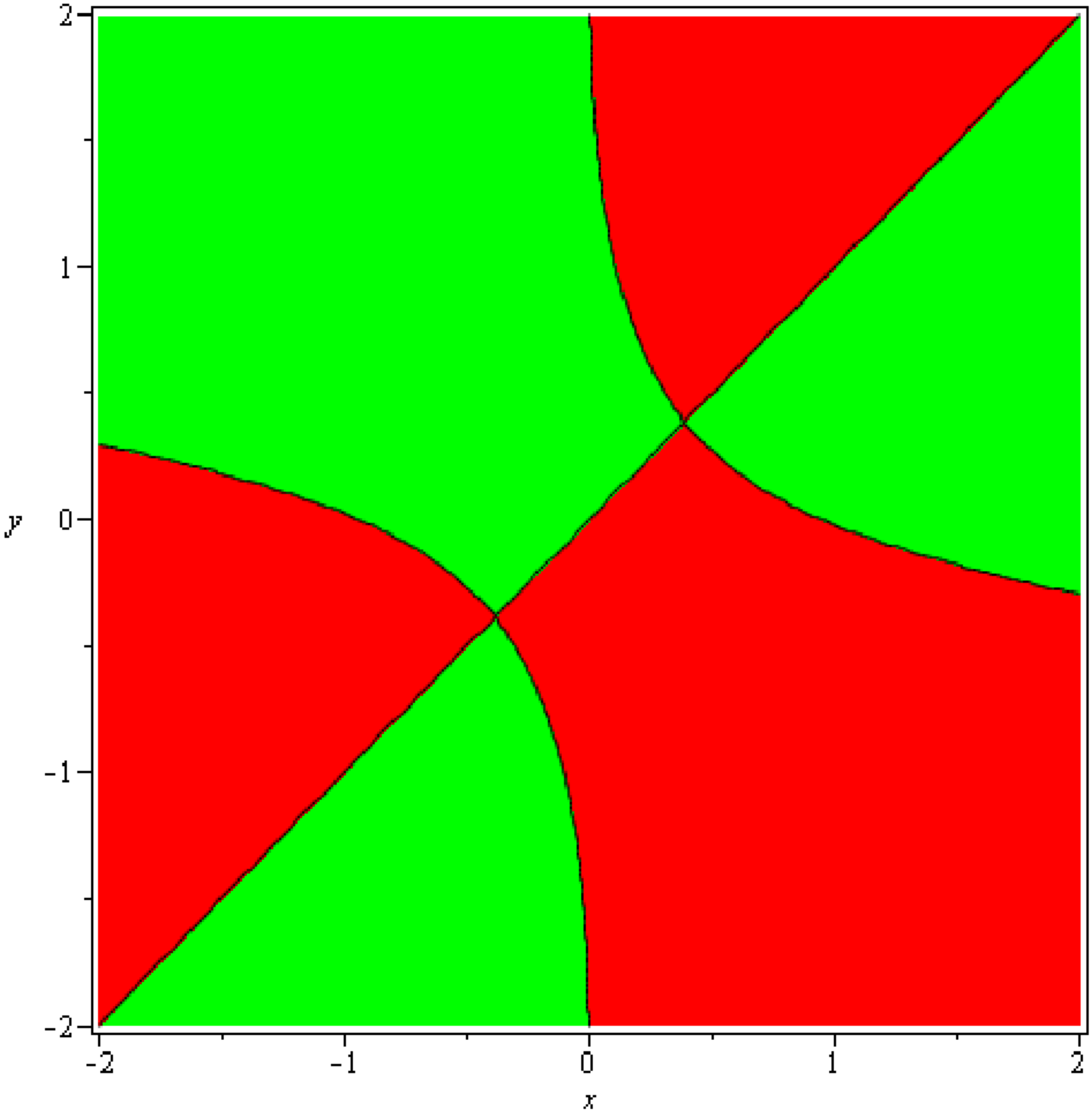
```
> Ds[y]:=(x,y)->2*alpha*(y-x)*exp(-alpha*(y-x)^2)*(-exp(-x^2)+exp
(-y^2)*exp(-alpha*(x-y)^2))/(-1+exp(-alpha*(x-y)^2)*exp(-alpha*
(y-x)^2))/exp(-y^2)-2*(exp(-alpha*(y-x)^2)*(-exp(-x^2)+exp(-y^2)
*exp(-alpha*(x-y)^2))/(-1+exp(-alpha*(x-y)^2)*exp(-alpha*(y-x)
^2))-exp(0)/(-1+exp(-alpha*(x-y)^2)*exp(-alpha*(y-x)^2))*(-exp(-
x^2)*exp(-alpha*(y-x)^2)+exp(-y^2)))/exp(-y^2)*y; #the selection
gradient in y-direction
```

$$Ds_y := (x, y) \;/\; \frac{2\,a\,(y\,K\,x)\,e^{K a\,(y\,K\,x)^2}\left(K e^{K x^2}\,C\,e^{K y^2}\,e^{K a\,(x\,K\,y)^2}\right)}{\left(K 1\,C\,e^{K a\,(x\,K\,y)^2}\,e^{K a\,(y\,K\,x)^2}\right)e^{K y^2}} \\ K\,\frac{2\left(\frac{e^{K a\,(y\,K\,x)^2}\left(K e^{K x^2}\,C\,e^{K y^2}\,e^{K a\,(x\,K\,y)^2}\right)}{K 1\,C\,e^{K a\,(x\,K\,y)^2}\,e^{K a\,(y\,K\,x)^2}}\,K\,\frac{e^{0}\left(K e^{K x^2}\,e^{K a\,(y\,K\,x)^2}\,C\,e^{K y^2}\right)}{K 1\,C\,e^{K a\,(x\,K\,y)^2}\,e^{K a\,(y\,K\,x)^2}}\right)y}{e^{K y^2}} \tag{29}$$

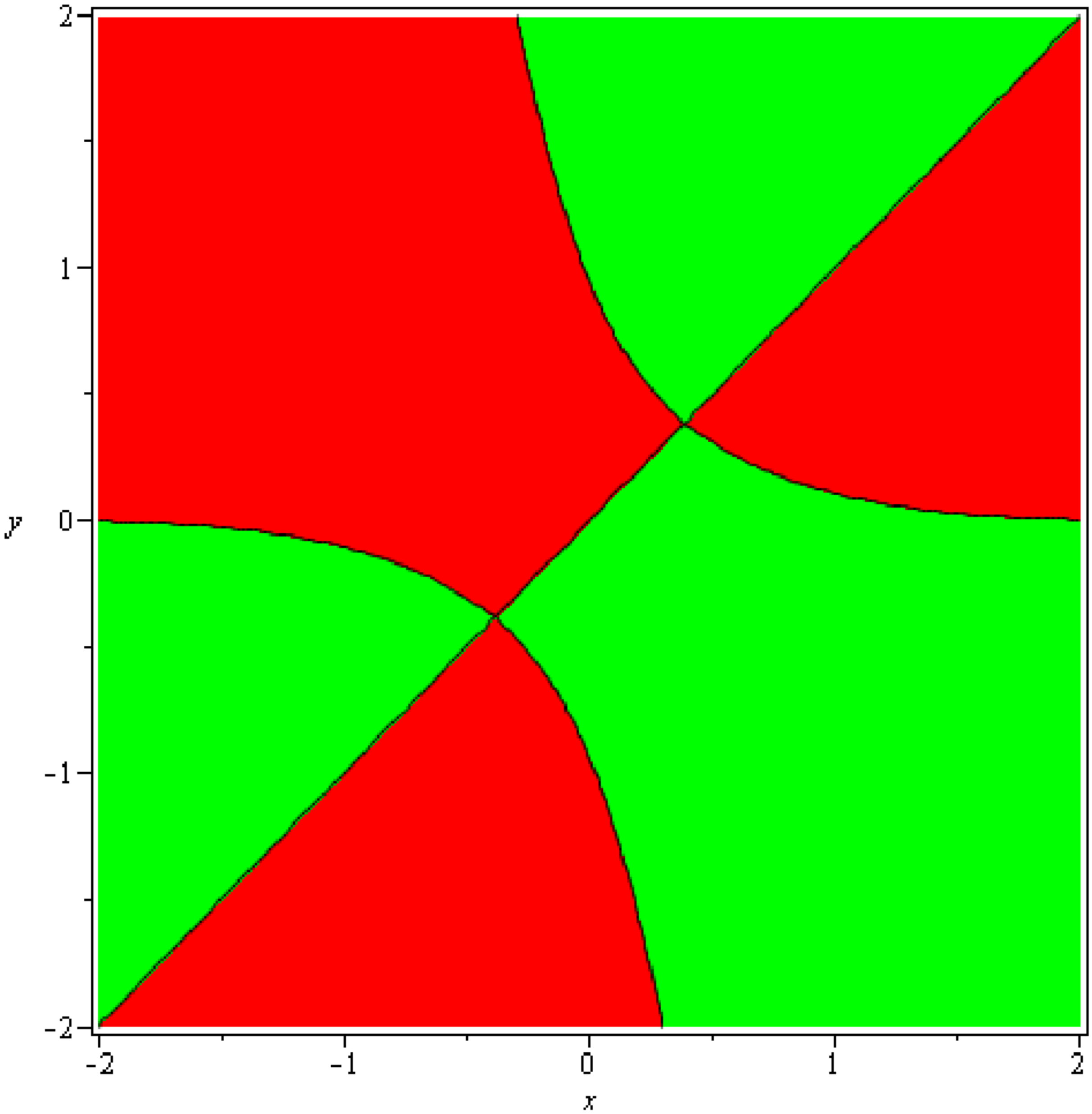
```
> with(plots):
> alpha:=0.7;
```

$$a := 0.7 \tag{30}$$

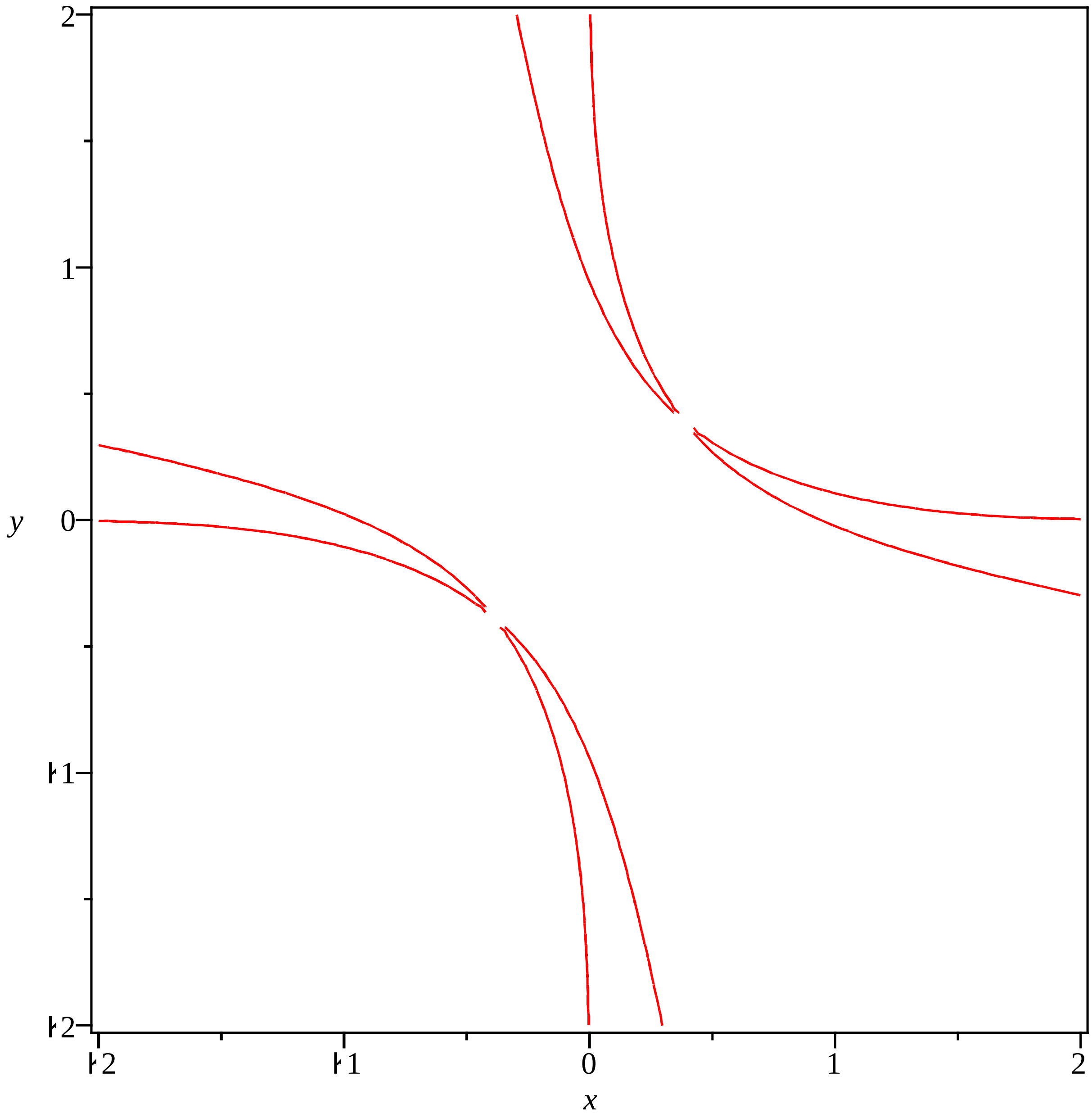
```
> contourplot(piecewise(abs(x-y)<0.01,0,Ds[x](x,y)),x=-2..2,y=-2.
.2,contours=[0],filled=true,coloring=[red,green],grid=[100,100],
axes=boxed);
```



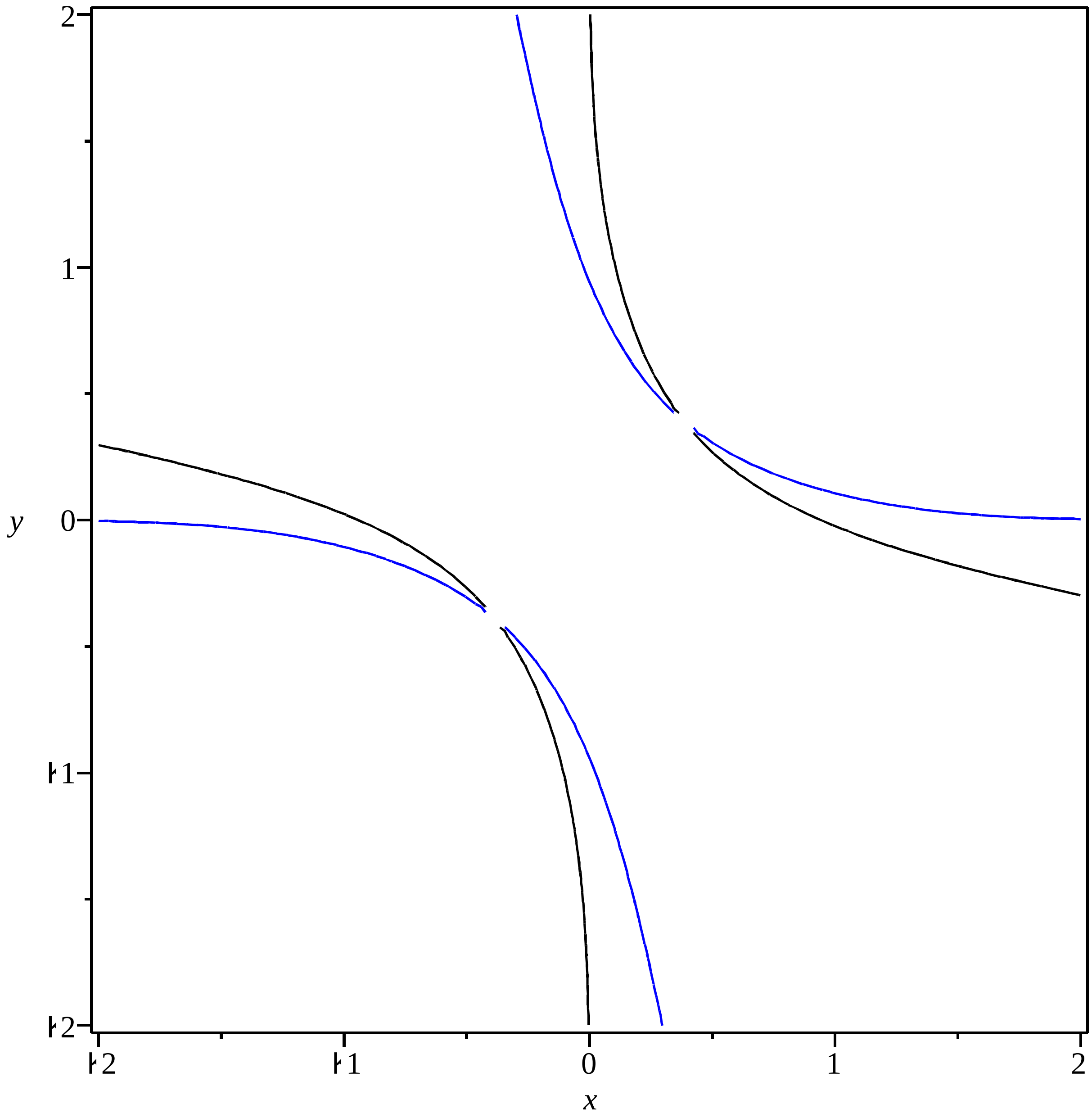

```
contourplot(piecewise(abs(x-y)<0.01,0,Ds[y](x,y)),x=-2..2,y=-2.
.2,contours=[0],filled=true,coloring=[red,green],grid=[100,100],
axes=boxed);
```




```
> contourplot({Ds[x](x,y),Ds[y](x,y)},x=-2..2,y=-2..2,contours=
  [0],grid=[100,100],axes=boxed);
```



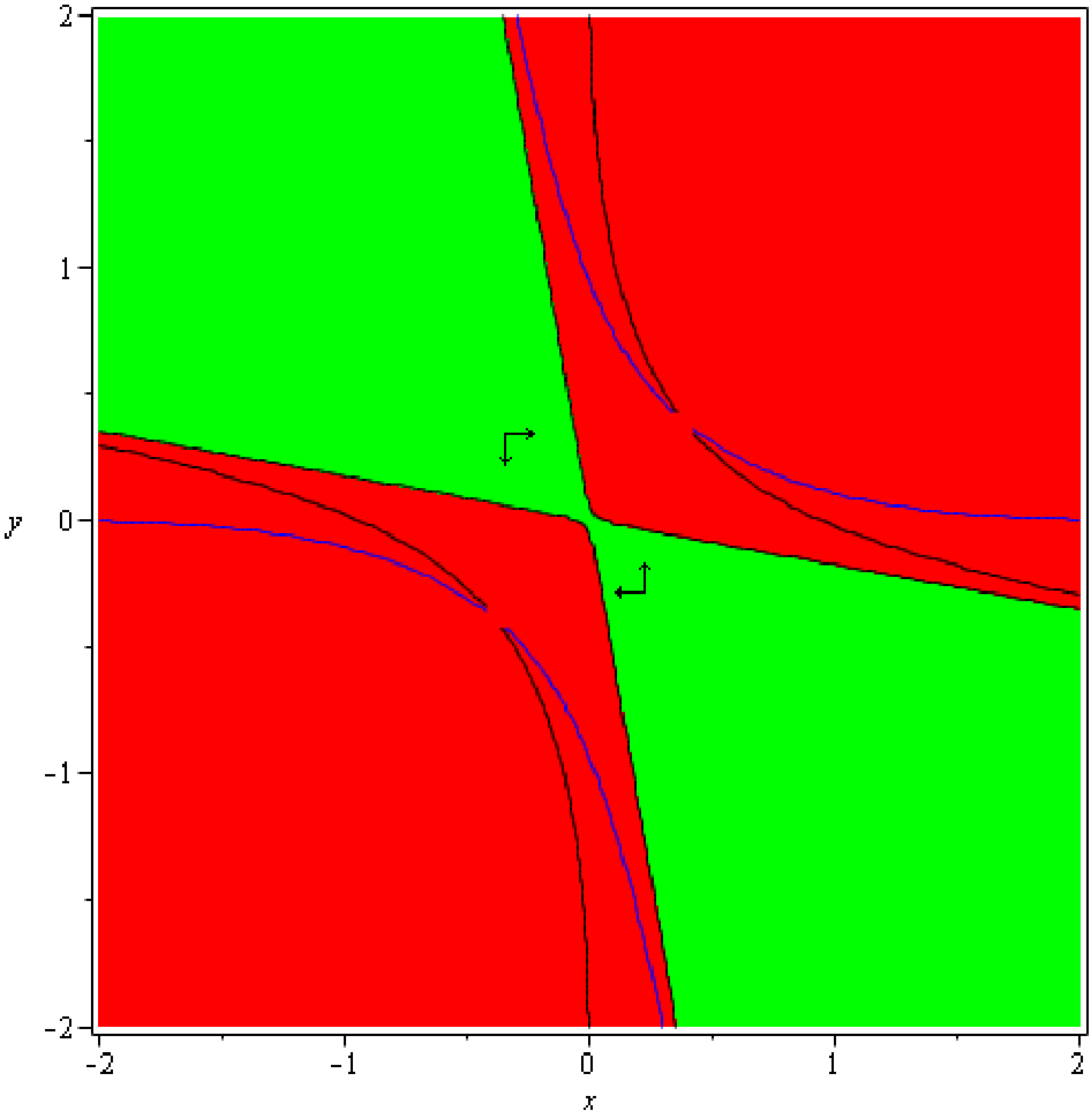

```
A:=contourplot(Ds[x](x,y),x=-2..2,y=-2..2,contours=[0],color=
black,grid=[100,100],axes=boxed):
B:=contourplot(Ds[y](x,y),x=-2..2,y=-2..2,contours=[0],color=
blue,grid=[100,100],axes=boxed):
display({A,B});
```




```
> s_mono:=(x,y)->r(y)*(1-(a(y,x)*K(x))/K(y));
```

$$s\_mono := (x, y) \rightarrow r(y)\left(1 - \frac{a(y, x)\, K(x)}{K(y)}\right) \quad (31)$$

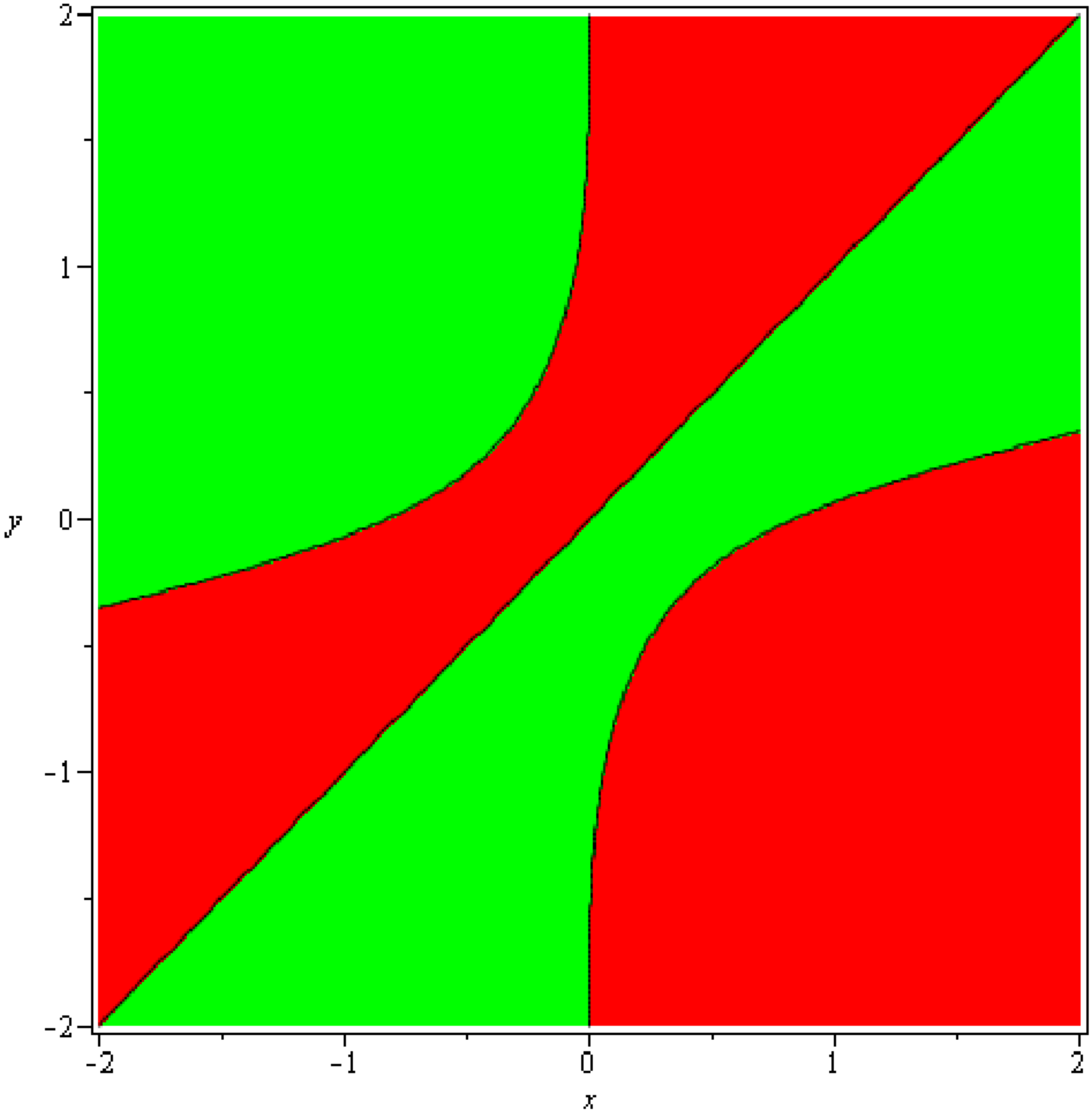
```
> C:=contourplot(s_mono(x,y)*s_mono(y,x),x=-2..2,y=-2..2,contours=
  [0],filled=true,coloring=[red,green],grid=[100,100],axes=boxed):
  #MIP
> display({A,B,C}); #no branching, the singularity is an ESS
  (which we aleady knew from the PIP :)
```



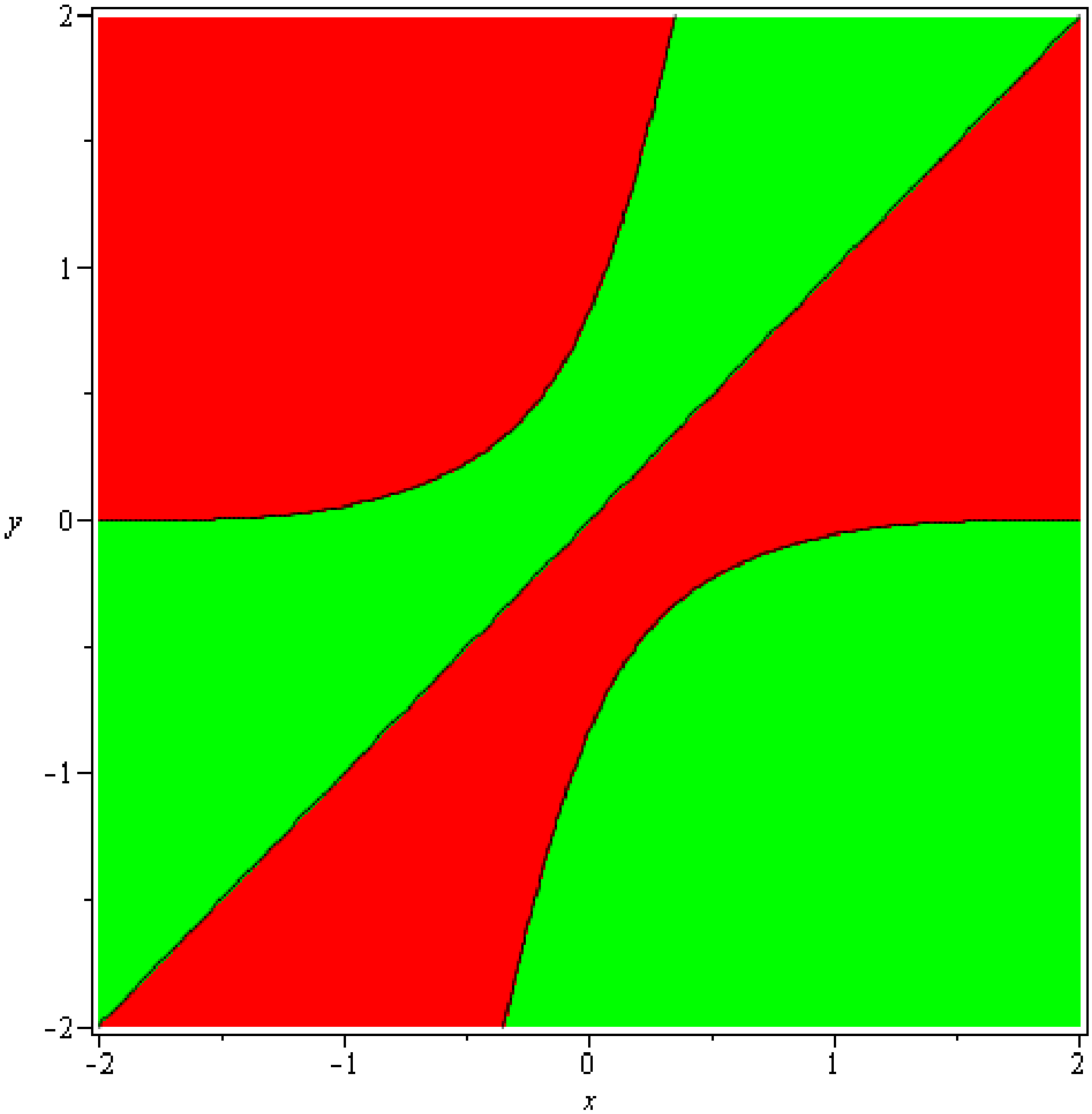

```
alpha:=1.5;
```

$$a := 1.5 \tag{32}$$

```
contourplot(piecewise(abs(x-y)<0.01,0,Ds[x](x,y)),x=-2..2,y=-2.
.2,contours=[0],filled=true,coloring=[red,green],grid=[100,100],
axes=boxed);
```




```
contourplot(piecewise(abs(x-y)<0.01,0,Ds[y](x,y)),x=-2..2,y=-2.
.2,contours=[0],filled=true,coloring=[red,green],grid=[100,100],
axes=boxed);
```



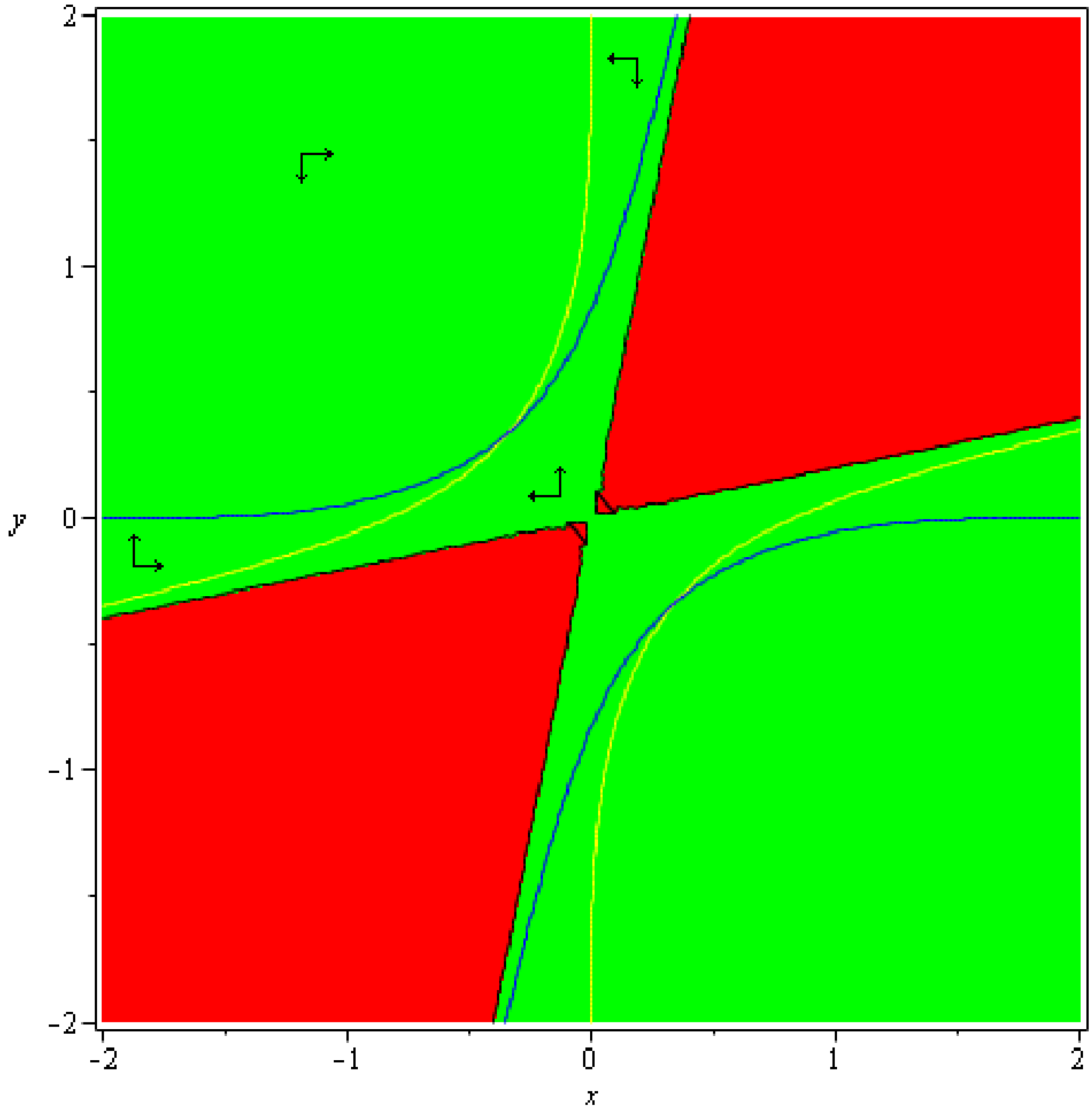

```
E:=contourplot(Ds[x](x,y),x=-2..2,y=-2..2,contours=[0],color=
yellow,grid=[100,100],axes=boxed):
```

```
F:=contourplot(Ds[y](x,y),x=-2..2,y=-2..2,contours=[0],color=
blue,grid=[100,100],axes=boxed):
```

```
G:=contourplot(s_mono(x,y)*s_mono(y,x),x=-2..2,y=-2..2,contours=
[0],grid=[100,100],axes=boxed):
```

```
G2:=contourplot(s_mono(x,y)*s_mono(y,x),x=-2..2,y=-2..2,
contours=[0], filled=true, coloring=[red,green], grid=[100,100],
axes=boxed):
```

```
display({E,F,G2}); #the monomorphic singularity is a branching
point, the population becomes dimorphic and evolves to the
dimorphic singularity
```




```
> subs(z=x,diff(s(x,y,z),z$2));
```

$$-\frac{1}{e^{-x^2}}\left(-\frac{3.0\,e^{-0.}\left(-1.\,e^{-x^2}+e^{-y^2}\,e^{-1.5\,(x-y)^2}\right)}{-1.+e^{-1.5\,(x-y)^2}\,e^{-1.5\,(y-x)^2}}+\frac{3.0\,e^{-1.5\,(x-y)^2}\left(-e^{-x^2}\,e^{-1.5\,(y-x)^2}+1.\,e^{-y^2}\right)}{-1.+e^{-1.5\,(x-y)^2}\,e^{-1.5\,(y-x)^2}}-\frac{(-3.0\,x+3.0\,y)^2\,e^{-1.5\,(x-y)^2}\left(-e^{-x^2}\,e^{-1.5\,(y-x)^2}+1.\,e^{-y^2}\right)}{-1.+e^{-1.5\,(x-y)^2}\,e^{-1.5\,(y-x)^2}}\right)+\frac{4\,(-3.0\,x+3.0\,y)\,e^{-1.5\,(x-y)^2}\left(-e^{-x^2}\,e^{-1.5\,(y-x)^2}+1.\,e^{-y^2}\right)x}{\left(-1.+e^{-1.5\,(x-y)^2}\,e^{-1.5\,(y-x)^2}\right)e^{-x^2}} \quad (33)$$

$$-\frac{1}{e^{-x^2}}\left(4\left(\frac{e^{-0.}\left(-1.\,e^{-x^2}+e^{-y^2}\,e^{-1.5\,(x-y)^2}\right)}{-1.+e^{-1.5\,(x-y)^2}\,e^{-1.5\,(y-x)^2}}-\frac{e^{-1.5\,(x-y)^2}\left(-e^{-x^2}\,e^{-1.5\,(y-x)^2}+1.\,e^{-y^2}\right)}{-1.+e^{-1.5\,(x-y)^2}\,e^{-1.5\,(y-x)^2}}\right)x^2\right)$$

$$-\frac{2\left(\frac{e^{-0.}\left(-1.\,e^{-x^2}+e^{-y^2}\,e^{-1.5\,(x-y)^2}\right)}{-1.+e^{-1.5\,(x-y)^2}\,e^{-1.5\,(y-x)^2}}-\frac{e^{-1.5\,(x-y)^2}\left(-e^{-x^2}\,e^{-1.5\,(y-x)^2}+1.\,e^{-y^2}\right)}{-1.+e^{-1.5\,(x-y)^2}\,e^{-1.5\,(y-x)^2}}\right)}{e^{-x^2}}$$

```
D2s[x]:=(x,y)->-(-3.0*exp(-0.)*(-1.*exp(-x^2)+exp(-y^2)*exp
(-1.5*(x-y)^2))/(-1.+exp(-1.5*(x-y)^2)*exp(-1.5*(y-x)^2))+3.0*
exp(-1.5*(x-y)^2)/(-1.+exp(-1.5*(x-y)^2)*exp(-1.5*(y-x)^2))*(-
exp(-x^2)*exp(-1.5*(y-x)^2)+1.*exp(-y^2))-(-3.0*x+3.0*y)^2*exp
(-1.5*(x-y)^2)/(-1.+exp(-1.5*(x-y)^2)*exp(-1.5*(y-x)^2))*(-exp(-
x^2)*exp(-1.5*(y-x)^2)+1.*exp(-y^2)))/exp(-x^2)+4*(-3.0*x+3.0*y)
*exp(-1.5*(x-y)^2)/(-1.+exp(-1.5*(x-y)^2)*exp(-1.5*(y-x)^2))*(-
exp(-x^2)*exp(-1.5*(y-x)^2)+1.*exp(-y^2))/exp(-x^2)*x-4*(exp(-0.
)*(-1.*exp(-x^2)+exp(-y^2)*exp(-1.5*(x-y)^2))/(-1.+exp(-1.5*(x-
y)^2)*exp(-1.5*(y-x)^2))-exp(-1.5*(x-y)^2)/(-1.+exp(-1.5*(x-y)
^2)*exp(-1.5*(y-x)^2))*(-exp(-x^2)*exp(-1.5*(y-x)^2)+1.*exp(-
y^2)))/exp(-x^2)*x^2-2*(exp(-0.)*(-1.*exp(-x^2)+exp(-y^2)*exp
(-1.5*(x-y)^2))/(-1.+exp(-1.5*(x-y)^2)*exp(-1.5*(y-x)^2))-exp
(-1.5*(x-y)^2)/(-1.+exp(-1.5*(x-y)^2)*exp(-1.5*(y-x)^2))*(-exp(-
x^2)*exp(-1.5*(y-x)^2)+1.*exp(-y^2)))/exp(-x^2); #the second
derivative of s(x,y,z) with respect to z evaluated at z=x, we
use this to check whether the dimorphic equilibrium allows for
further branching or not
```

$$D2s_x := (x,y) \to -\frac{1}{e^{-x^2}}\left(-\frac{3.0\,e^{-0.}\left(-1.\,e^{-x^2}+e^{-y^2}\,e^{-1.5\,(x-y)^2}\right)}{-1.+e^{-1.5\,(x-y)^2}\,e^{-1.5\,(y-x)^2}}\right. \tag{34}$$

$$+\frac{3.0\,e^{-1.5\,(x-y)^2}\left(-e^{-x^2}\,e^{-1.5\,(y-x)^2}+1.\,e^{-y^2}\right)}{-1.+e^{-1.5\,(x-y)^2}\,e^{-1.5\,(y-x)^2}}$$

$$\left.-\frac{(-3.0\,x+3.0\,y)^2\,e^{-1.5\,(x-y)^2}\left(-e^{-x^2}\,e^{-1.5\,(y-x)^2}+1.\,e^{-y^2}\right)}{-1.+e^{-1.5\,(x-y)^2}\,e^{-1.5\,(y-x)^2}}\right)$$

$$+\frac{4\,(-3.0\,x+3.0\,y)\,e^{-1.5\,(x-y)^2}\left(-e^{-x^2}\,e^{-1.5\,(y-x)^2}+1.\,e^{-y^2}\right)x}{\left(-1.+e^{-1.5\,(x-y)^2}\,e^{-1.5\,(y-x)^2}\right)e^{-x^2}}$$

$$-\frac{1}{e^{-x^2}}\left(4\left(\frac{e^{-0.}\left(-1.\,e^{-x^2}+e^{-y^2}\,e^{-1.5\,(x-y)^2}\right)}{-1.+e^{-1.5\,(x-y)^2}\,e^{-1.5\,(y-x)^2}}\right.\right.$$

$$K \frac{e^{ķ1.5\,(x\,K\,y)^2}\left(ķe^{ķx^2}\,e^{ķ1.5\,(y\,K\,x)^2} C\,1.\,e^{ķy^2}\right)}{ķ1.\,C\,e^{ķ1.5\,(x\,K\,y)^2}\,e^{ķ1.5\,(y\,K\,x)^2}}\Bigg)\,x^2\Bigg)$$

$$K \frac{2\left(\frac{e^{ķ0.}\left(ķ1.\,e^{ķx^2} C\,e^{ķy^2}\,e^{ķ1.5\,(x\,K\,y)^2}\right)}{ķ1.\,C\,e^{ķ1.5\,(x\,K\,y)^2}\,e^{ķ1.5\,(y\,K\,x)^2}} K \frac{e^{ķ1.5\,(x\,K\,y)^2}\left(ķe^{ķx^2}\,e^{ķ1.5\,(y\,K\,x)^2} C\,1.\,e^{ķy^2}\right)}{ķ1.\,C\,e^{ķ1.5\,(x\,K\,y)^2}\,e^{ķ1.5\,(y\,K\,x)^2}}\right)}{e^{ķx^2}}$$

```
O subs(z=y,diff(s(x,y,z),z$2));
```

$$ķ\frac{1}{e^{ķy^2}}\Bigg(ķ\frac{3.0\,e^{ķ1.5\,(y\,K\,x)^2}\left(ķ1.\,e^{ķx^2} C\,e^{ķy^2}\,e^{ķ1.5\,(x\,K\,y)^2}\right)}{ķ1.\,C\,e^{ķ1.5\,(x\,K\,y)^2}\,e^{ķ1.5\,(y\,K\,x)^2}} \qquad \textbf{(35)}$$

$$C \frac{(ķ3.0\,y\,C\,3.0\,x)^2\,e^{ķ1.5\,(y\,K\,x)^2}\left(ķ1.\,e^{ķx^2} C\,e^{ķy^2}\,e^{ķ1.5\,(x\,K\,y)^2}\right)}{ķ1.\,C\,e^{ķ1.5\,(x\,K\,y)^2}\,e^{ķ1.5\,(y\,K\,x)^2}}$$

$$C \frac{3.0\,e^{ķ0.}\left(ķe^{ķx^2}\,e^{ķ1.5\,(y\,K\,x)^2} C\,1.\,e^{ķy^2}\right)}{ķ1.\,C\,e^{ķ1.5\,(x\,K\,y)^2}\,e^{ķ1.5\,(y\,K\,x)^2}}\Bigg)$$

$$K \frac{4\,(ķ3.0\,y\,C\,3.0\,x)\,e^{ķ1.5\,(y\,K\,x)^2}\left(ķ1.\,e^{ķx^2} C\,e^{ķy^2}\,e^{ķ1.5\,(x\,K\,y)^2}\right)y}{\left(ķ1.\,C\,e^{ķ1.5\,(x\,K\,y)^2}\,e^{ķ1.5\,(y\,K\,x)^2}\right)e^{ķy^2}}$$

$$K \frac{1}{e^{ķy^2}}\Bigg(4\Bigg(\frac{e^{ķ1.5\,(y\,K\,x)^2}\left(ķ1.\,e^{ķx^2} C\,e^{ķy^2}\,e^{ķ1.5\,(x\,K\,y)^2}\right)}{ķ1.\,C\,e^{ķ1.5\,(x\,K\,y)^2}\,e^{ķ1.5\,(y\,K\,x)^2}}$$

$$K \frac{e^{ķ0.}\left(ķe^{ķx^2}\,e^{ķ1.5\,(y\,K\,x)^2} C\,1.\,e^{ķy^2}\right)}{ķ1.\,C\,e^{ķ1.5\,(x\,K\,y)^2}\,e^{ķ1.5\,(y\,K\,x)^2}}\Bigg)\,y^2\Bigg)$$

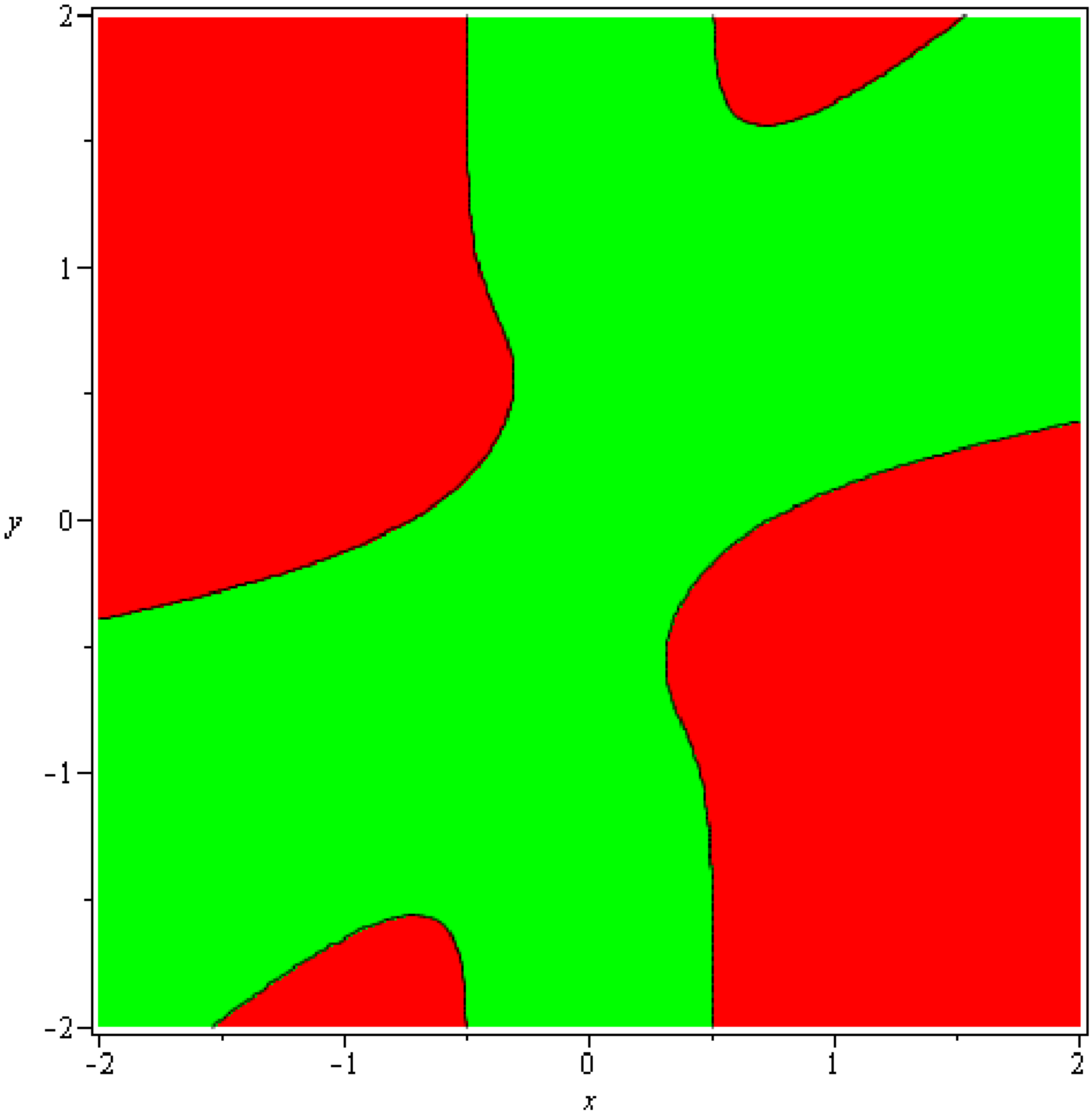
$$K \frac{2\left(\frac{e^{ķ1.5\,(y\,K\,x)^2}\left(ķ1.\,e^{ķx^2} C\,e^{ķy^2}\,e^{ķ1.5\,(x\,K\,y)^2}\right)}{ķ1.\,C\,e^{ķ1.5\,(x\,K\,y)^2}\,e^{ķ1.5\,(y\,K\,x)^2}} K \frac{e^{ķ0.}\left(ķe^{ķx^2}\,e^{ķ1.5\,(y\,K\,x)^2} C\,1.\,e^{ķy^2}\right)}{ķ1.\,C\,e^{ķ1.5\,(x\,K\,y)^2}\,e^{ķ1.5\,(y\,K\,x)^2}}\right)}{e^{ķy^2}}$$

```
O D2s[y]:=(x,y)->-(-3.0*exp(-1.5*(y-x)^2)*(-1.*exp(-x^2)+exp(-y^2)
  *exp(-1.5*(x-y)^2))/(-1.+exp(-1.5*(x-y)^2)*exp(-1.5*(y-x)^2))+
  (-3.0*y+3.0*x)^2*exp(-1.5*(y-x)^2)*(-1.*exp(-x^2)+exp(-y^2)*exp
  (-1.5*(x-y)^2))/(-1.+exp(-1.5*(x-y)^2)*exp(-1.5*(y-x)^2))+3.0*
  exp(-0.)/(-1.+exp(-1.5*(x-y)^2)*exp(-1.5*(y-x)^2))*(-exp(-x^2)*
  exp(-1.5*(y-x)^2)+1.*exp(-y^2)))/exp(-y^2)-4*(-3.0*y+3.0*x)*exp
  (-1.5*(y-x)^2)*(-1.*exp(-x^2)+exp(-y^2)*exp(-1.5*(x-y)^2))/(-1.+
  exp(-1.5*(x-y)^2)*exp(-1.5*(y-x)^2))/exp(-y^2)*y-4*(exp(-1.5*(y-
  x)^2)*(-1.*exp(-x^2)+exp(-y^2)*exp(-1.5*(x-y)^2))/(-1.+exp(-1.5*
  (x-y)^2)*exp(-1.5*(y-x)^2))-exp(-0.)/(-1.+exp(-1.5*(x-y)^2)*exp
  (-1.5*(y-x)^2))*(-exp(-x^2)*exp(-1.5*(y-x)^2)+1.*exp(-y^2)))/exp
  (-y^2)*y^2-2*(exp(-1.5*(y-x)^2)*(-1.*exp(-x^2)+exp(-y^2)*exp
```

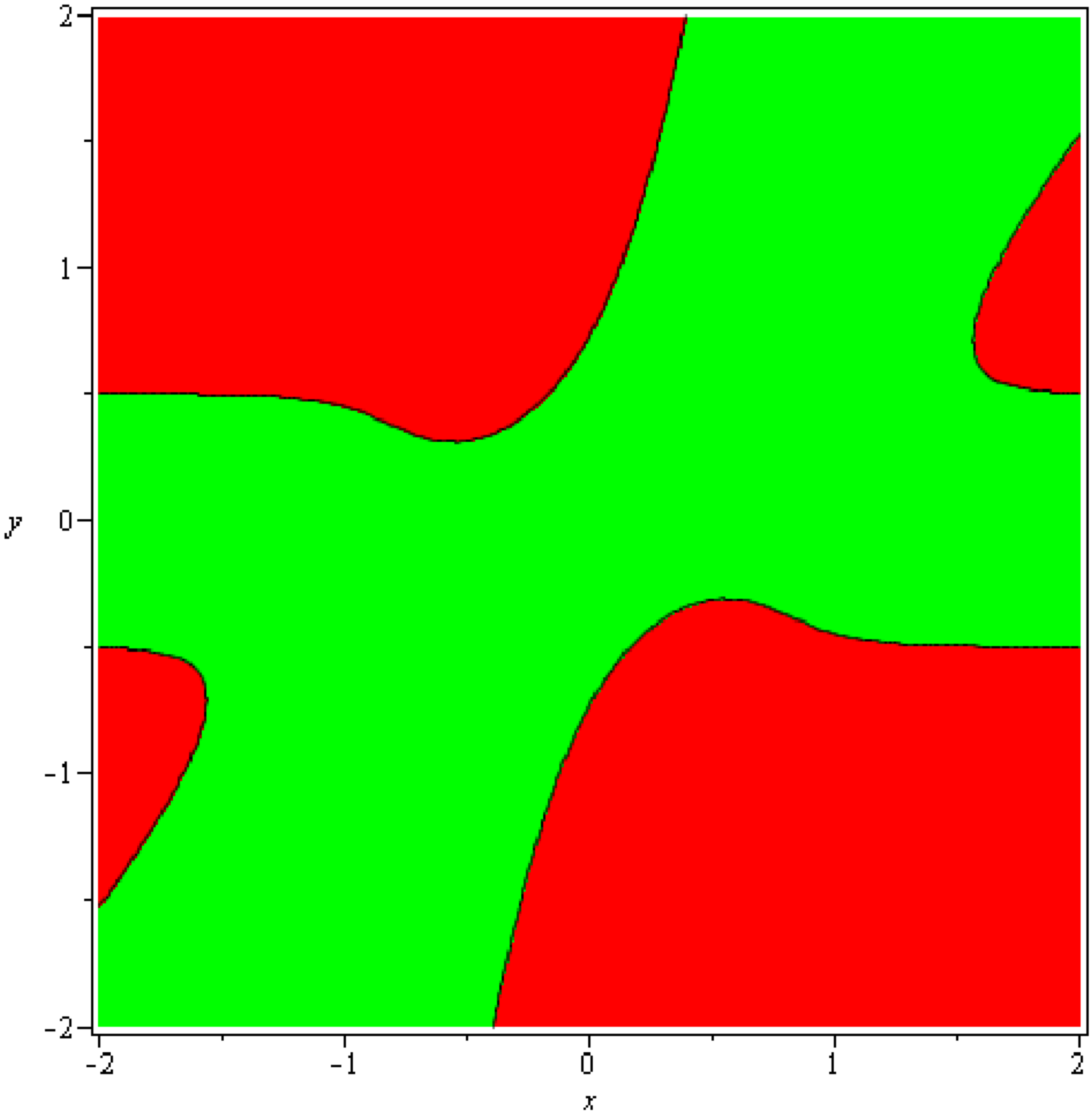
```
(-1.5*(x-y)^2))/(-1.+exp(-1.5*(x-y)^2)*exp(-1.5*(y-x)^2))-exp
(-0.)/(-1.+exp(-1.5*(x-y)^2)*exp(-1.5*(y-x)^2))*(-exp(-x^2)*exp
(-1.5*(y-x)^2)+1.*exp(-y^2)))/exp(-y^2);
```

$$D2s_y := (x, y) \to -\frac{1}{e^{-y^2}}\left(-\frac{3.0\, e^{-1.5\,(y-x)^2}\left(-1.\, e^{-x^2}+e^{-y^2}\, e^{-1.5\,(x-y)^2}\right)}{-1.+e^{-1.5\,(x-y)^2}\, e^{-1.5\,(y-x)^2}} + \frac{(-3.0\,y+3.0\,x)^2\, e^{-1.5\,(y-x)^2}\left(-1.\, e^{-x^2}+e^{-y^2}\, e^{-1.5\,(x-y)^2}\right)}{-1.+e^{-1.5\,(x-y)^2}\, e^{-1.5\,(y-x)^2}} + \frac{3.0\, e^{-0.}\left(-e^{-x^2}\, e^{-1.5\,(y-x)^2}+1.\, e^{-y^2}\right)}{-1.+e^{-1.5\,(x-y)^2}\, e^{-1.5\,(y-x)^2}}\right) - \frac{4\,(-3.0\,y+3.0\,x)\, e^{-1.5\,(y-x)^2}\left(-1.\, e^{-x^2}+e^{-y^2}\, e^{-1.5\,(x-y)^2}\right) y}{\left(-1.+e^{-1.5\,(x-y)^2}\, e^{-1.5\,(y-x)^2}\right) e^{-y^2}} - \frac{1}{e^{-y^2}}\left(4\left(\frac{e^{-1.5\,(y-x)^2}\left(-1.\, e^{-x^2}+e^{-y^2}\, e^{-1.5\,(x-y)^2}\right)}{-1.+e^{-1.5\,(x-y)^2}\, e^{-1.5\,(y-x)^2}} - \frac{e^{-0.}\left(-e^{-x^2}\, e^{-1.5\,(y-x)^2}+1.\, e^{-y^2}\right)}{-1.+e^{-1.5\,(x-y)^2}\, e^{-1.5\,(y-x)^2}}\right) y^2\right) - \frac{2\left(\frac{e^{-1.5\,(y-x)^2}\left(-1.\, e^{-x^2}+e^{-y^2}\, e^{-1.5\,(x-y)^2}\right)}{-1.+e^{-1.5\,(x-y)^2}\, e^{-1.5\,(y-x)^2}} - \frac{e^{-0.}\left(-e^{-x^2}\, e^{-1.5\,(y-x)^2}+1.\, e^{-y^2}\right)}{-1.+e^{-1.5\,(x-y)^2}\, e^{-1.5\,(y-x)^2}}\right)}{e^{-y^2}} \tag{36}$$

```
> contourplot(piecewise(abs(x-y)<0.01,0,D2s[x](x,y)),x=-2..2,y=-2.
.2,filled=true,contours=[0],coloring=[red,green],grid=[100,100],
axes=boxed);
```



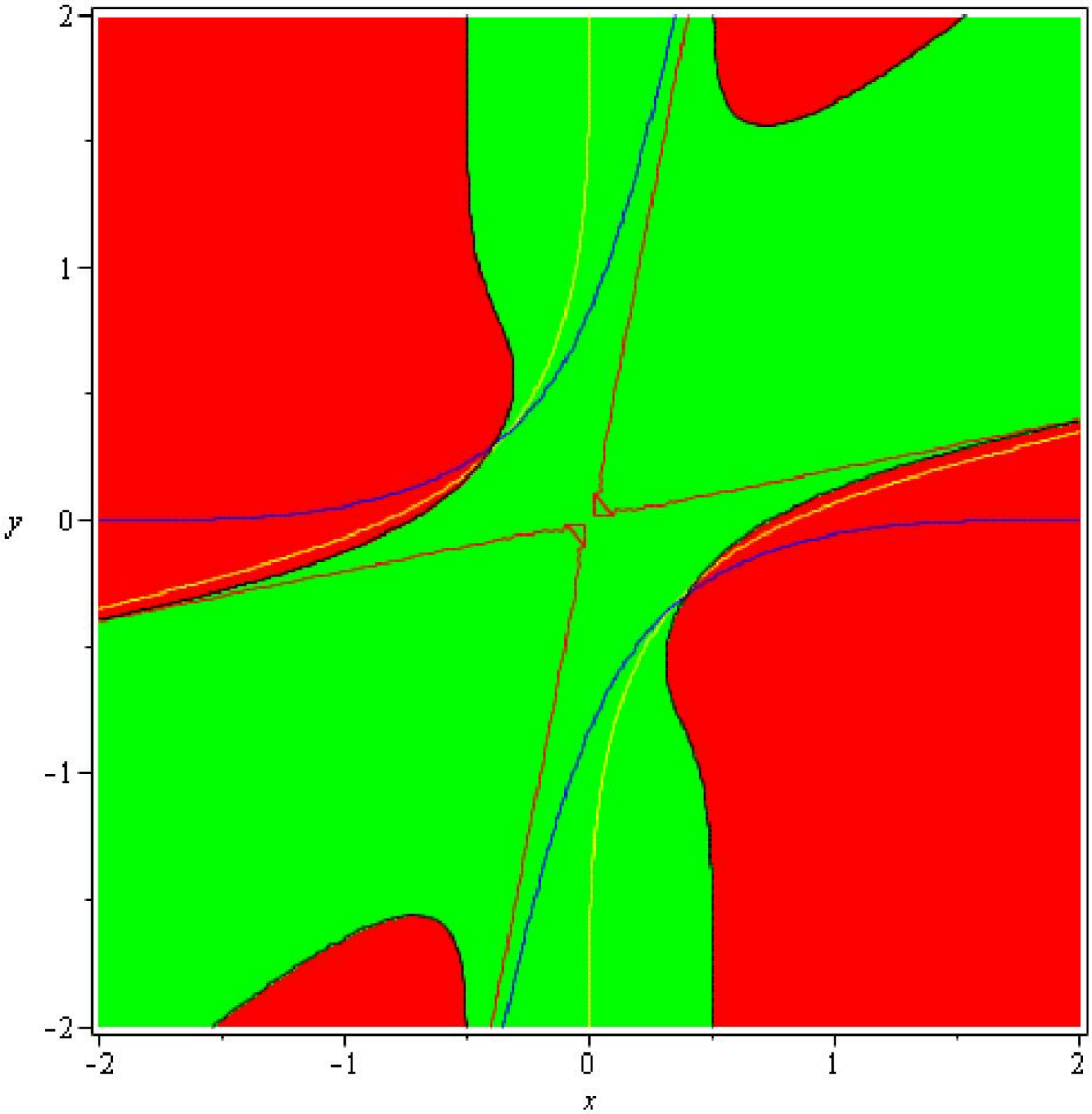

```
> contourplot(piecewise(abs(x-y)<0.01,0,D2s[y](x,y)),x=-2..2,y=-2..2,filled=true,contours=[0],coloring=[red,green],grid=[100,100],axes=boxed);
```



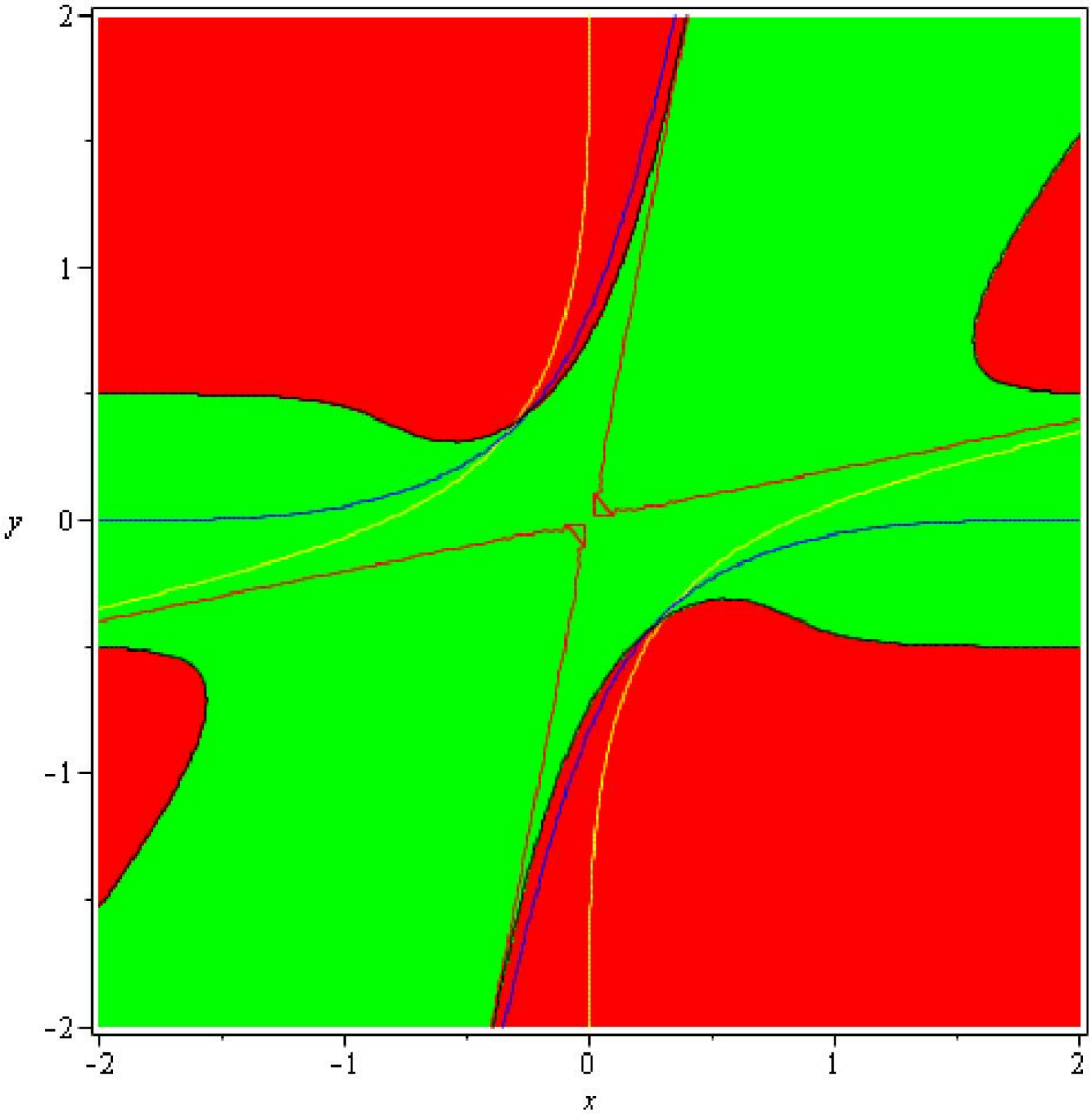

```
H:=contourplot(piecewise(abs(x-y)<0.01,0,D2s[x](x,y)),x=-2..2,y=
-2..2,contours=[0],filled, coloring=[red,green] ,grid=[100,100],
axes=boxed):
```

```
J:=contourplot(piecewise(abs(x-y)<0.01,0,D2s[y](x,y)),x=-2..2,y=
-2..2,contours=[0],filled=true, coloring=[red,green],grid=[100,
100],axes=boxed):
```

```
display({E,F,G,H});
```

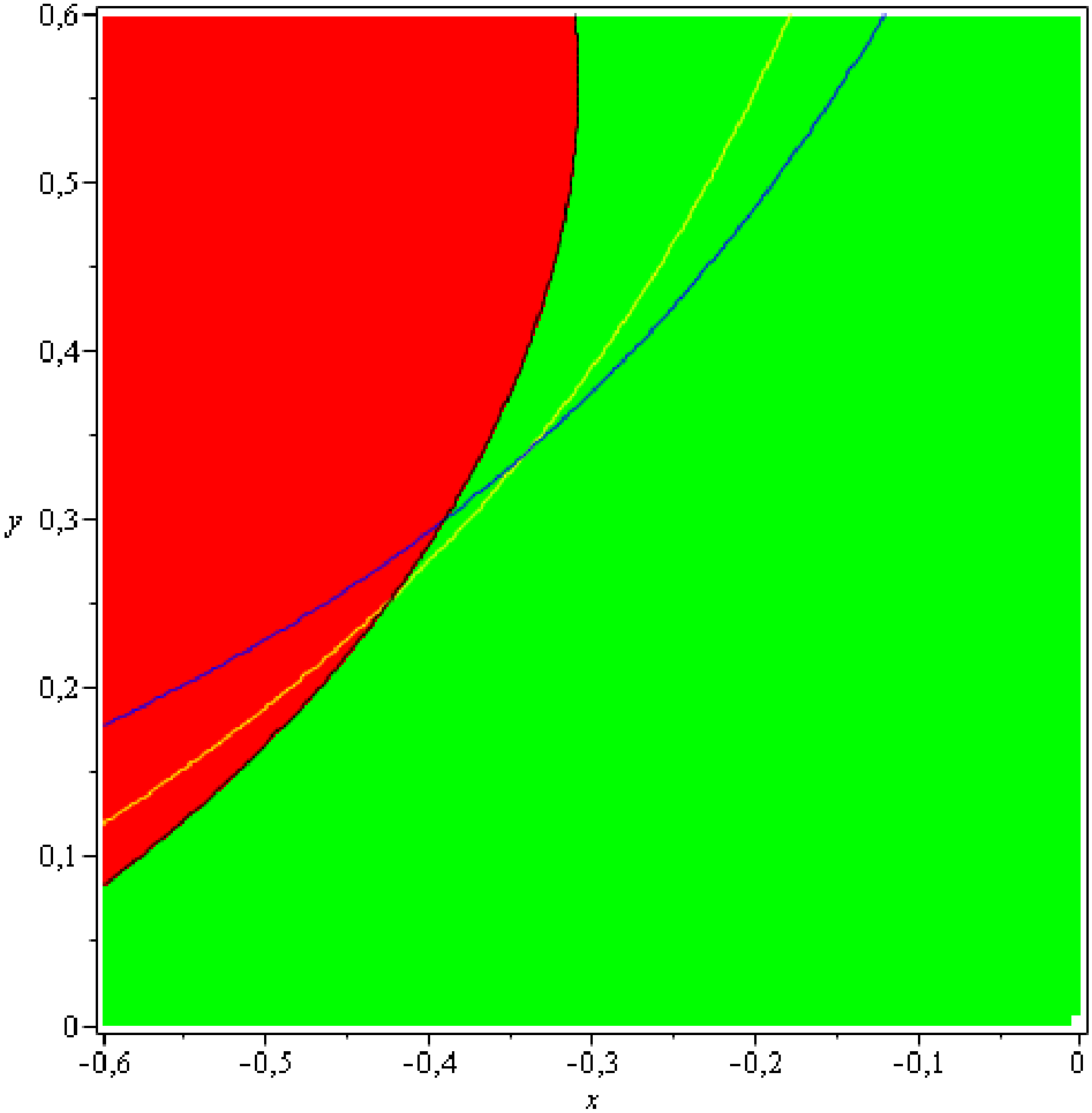


O display({E,F,G,J});

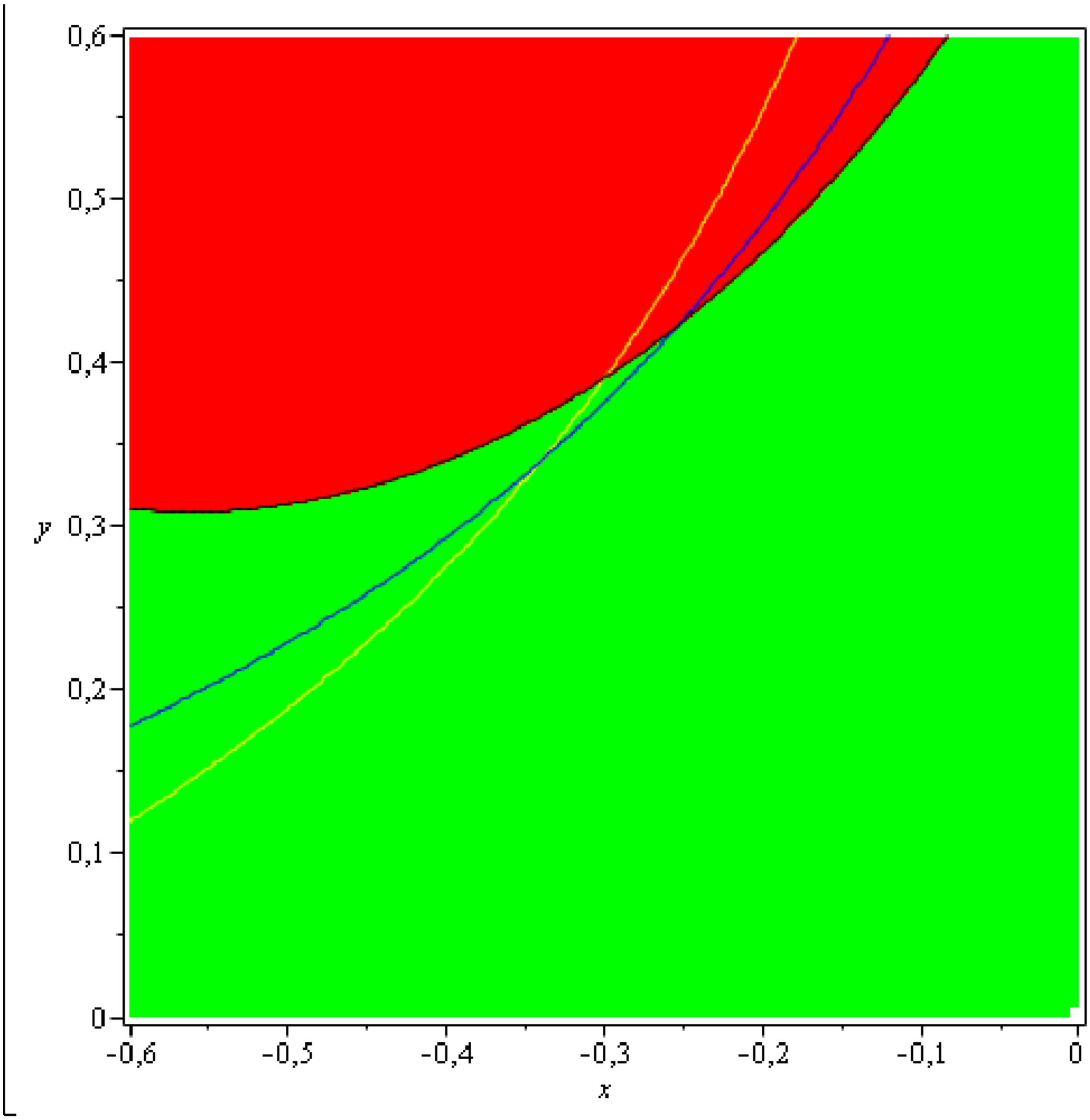



```
Ezoom:=contourplot(Ds[x](x,y),x=-0.6..0,y=0..0.6,contours=[0],
color=yellow,grid=[100,100],axes=boxed):
Fzoom:=contourplot(Ds[y](x,y),x=-0.6..0,y=0..0.6,contours=[0],
color=blue,grid=[100,100],axes=boxed):
Gzoom:=contourplot(s_mono(x,y)*s_mono(y,x),x=-0.6..0,y=0..0.6,
contours=[0],grid=[100,100],axes=boxed):
Hzoom:=contourplot(D2s[x](x,y),x=-0.6..0,y=0..0.6,contours=[0],
filled, coloring=[red,green] ,grid=[100,100],axes=boxed):
Jzoom:=contourplot(D2s[y](x,y),x=-0.6..0,y=0..0.6,contours=[0],
filled=true, coloring=[red,green],grid=[100,100],axes=boxed):
display({Ezoom,Fzoom,Gzoom,Hzoom}); #the dimorphic singularity
is invadable for small mutations in x -> branching into a
```

trimorphic population




```
display({Ezoom,Fzoom,Gzoom,Jzoom}); #the dimorphic singularity
is invadable for small mutations in y -> branching into a
trimorphic population
```



0,6
0,5
0,4
y 0,3
0,2
0,1
0
-0,6
-0,5
-0,4
-0,3
-0,2
-0,1
0
x