

II

Lotka-Volterra competition model (continued).

Dimorphic resident pop. dynamics.

$$\left\{ \begin{array}{l} \dot{n}_1 = r(x_1) n_1 \left(1 - \frac{n_1 + a(x_1, x_2) n_2}{K(x_1)} \right) \\ \dot{n}_2 = r(x_2) n_2 \left(1 - \frac{n_2 + a(x_2, x_1) n_1}{K(x_2)} \right) \end{array} \right.$$

If $S_{x_1}(x_2) > 0$ and $S_{x_2}(x_1) > 0$, then there exists a unique positive equilibrium.
(see phase plane analysis prev. lect.)

Equilibrium equations:

$$\dot{n}_1 = 0 \quad \& \quad \dot{n}_2 = 0$$

\Rightarrow pos. equil. (\hat{n}_1, \hat{n}_2) given by

$$\left\{ \begin{array}{l} \hat{n}_1(x_1, x_2) = \frac{K(x_1) - a(x_1, x_2) K(x_2)}{1 - a(x_1, x_2) a(x_2, x_1)} > 0 \\ \hat{n}_2(x_1, x_2) = \frac{K(x_2) - a(x_2, x_1) K(x_1)}{1 - a(x_1, x_2) a(x_2, x_1)} > 0 \end{array} \right.$$

Resident - mutant pop. dynamics.

$$\left\{ \begin{array}{l} \dot{n}_1 = r(x_1) n_1 \left(1 - \frac{n_1 + a(x_1, x_2) n_2 + a(x_1, y) m}{K(x_1)} \right) \\ \dot{n}_2 = r(x_2) n_2 \left(1 - \frac{n_2 + a(x_2, x_1) n_1 + a(x_2, y) m}{K(x_2)} \right) \\ \dot{m} = r(y) m \left(1 - \frac{m + a(y, x_1) n_1 + a(y, x_2) n_2}{K(y)} \right) \end{array} \right.$$

Invasion fitness

$$S_{x_1, x_2}(y) := \lim_{\substack{n_1 \rightarrow \hat{n}_1(x_1, x_2) \\ n_2 \rightarrow \hat{n}_2(x_1, x_2) \\ m \rightarrow 0}} \frac{\dot{m}}{m}$$

(i.e., the exponential growth rate of a small mutant population in a dimorphic resident population at its equilibrium)

$$\Rightarrow S_{x_1, x_2}(y) = r(y) \left(1 - \frac{a(y, x_1) \hat{n}_1(x_1, x_2) + a(y, x_2) \hat{n}_2(x_1, x_2)}{k(y)} \right)$$

Notice

The equilibrium equations $\dot{n}_1 = 0$ and $\dot{n}_2 = 0$ for the positive resident pop. equilibrium are equivalent to the equations

$$S_{x_1, x_2}(x_1) = 0 \quad \text{and} \quad S_{x_1, x_2}(x_2) = 0.$$

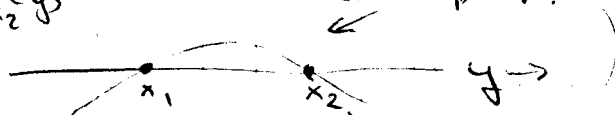
(i.e., the resident strategies x_1 and x_2 are selectively neutral among themselves.)

Fitness landscape

The fitness landscape is the graph of $S_{x_1, x_2}(y)$ of the invasion fitness as a function of the invader's strategy given the resident strategies:

$$\uparrow \\ S_{x_1, x_2}(y)$$

(Example)



What does the fitness landscape look like in a neighborhood of $x^* = 0$?

Claim

The fitness landscape is locally concave (resp. convex) in a neighborhood of $x^* = 0$ if $0 < \alpha < 1$ (resp. $\alpha > 1$).

Proof.

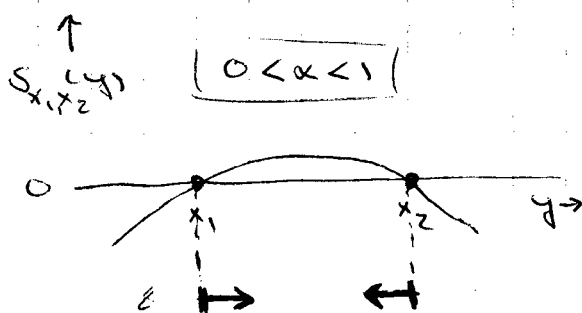
$$\partial_y^2 S_{x_1, x_2}(y) \rightarrow 2(\alpha - 1) \text{ as } y, x_1, x_2 \rightarrow x^*$$

By continuity, there exists a neighborhood U of x^* such that

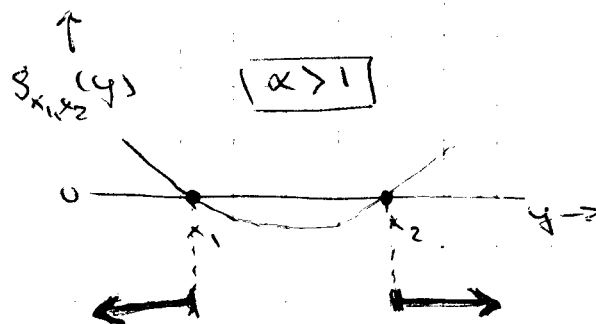
$$\text{sign } \partial_y^2 S_{x_1, x_2}(y) = \text{sign } 2(\alpha - 1) \quad \forall x_1, x_2, y \in U.$$

□

Conclusion



⇒ Only mutants with strategy in between the resident strategies can invade

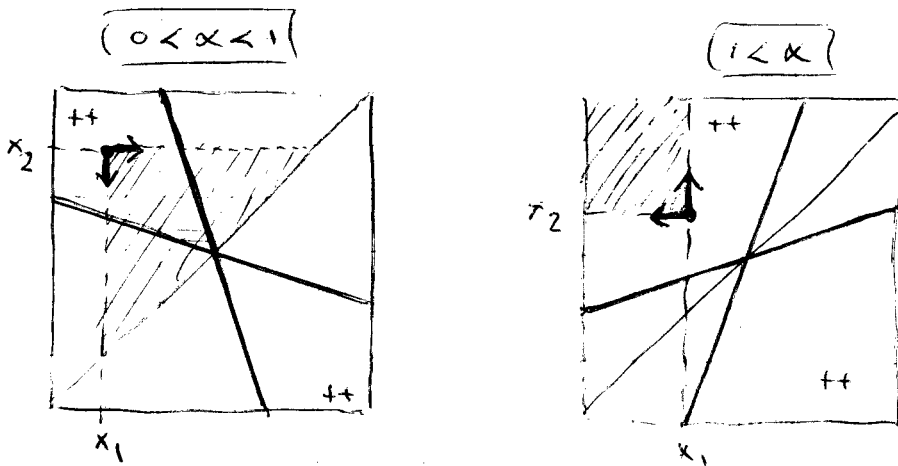


Only mutants with strategies outside the resident strategies can invade.

Invasion cone

4

We can represent the same info (i.e., who can invade a dimorphic resident pop.) in the area of mutual invadability:



So, now we know who can invade (and who cannot) once the population has become dimorphic in a neighborhood of $x^* = 0$.

But what happens after an invasion? Do the mutant and the two residents all three coexist? Or does the mutant "kick-out" one of the residents or both residents?

To answer these questions one has to study the full 3-dimensional dynamics of the mutant-resident₁-resident₂ dynamics.

There is no advantage doing this for the present LV comp. model. In fact, it is much easier to do it ~~for~~ for a more general class of models.

But, if we could prove that three similar strategies (x_1, x_2, x_3) in a neighborhood of $x^*=0$ cannot coexist, ...

(and we shall prove this later in a much more general context)

then we can conclude the following:

$0 < \alpha < 1$ \Rightarrow • The monomorphic pop. evolves towards $x^*=0$.

• Also the dimorphic pop. evolves towards $x^*=0$.

$1 < \alpha$ \Rightarrow • The monomorphic pop. evolves towards $x^*=0$.

• The dimorphic pop. evolves away from $x^*=0$.

Combining all we know (terminology)

$x^*=0$ is both monomorphically attracting as well as dimorphically attracting if $0 < \alpha < 1$.

But for $\alpha > 1$, we have that $x^*=0$ is monomorphically attracting, but dimorphically repelling.

That is why we get evolutionary branching for $\alpha > 1$, but not for $0 < \alpha < 1$.

(see simulation results in first lecture).