IV Differential inclusions and total stability.

Consider a sequence of invasion-substitution events in a k-morphic ($k \geq 1$) resident population.

The corresponding sequence of resident strategies is denoted by $\underline{x}^{(1)}, \underline{x}^{(2)}, \ldots$ where $\underline{x}^{(i)} = (x_1^{(i)}, \ldots, x_k^{(i)})$ is the vector of resident strategies directly after the i^{th} invasion-substitution event.

The sequence $\underline{x}^{(1)}, \underline{x}^{(2)}, \ldots$ is called a ($\underline{k\text{-morphic}}$) $\underline{\text{substitution sequence}}$

If $\underline{x}^{(1)}, \underline{x}^{(2)}, \ldots$ is a substitution sequence, then necessarily

(1a) $S_{\underline{x}^{(i)}}(x_j^{(i+1)}) \geq 0 \quad \forall i, j$ and

(1b) $S_{\underline{x}^{(i+1)}}(x_j^{(i)}) \leq 0 \quad \forall i, j$

Note, that because of low mutation rates, we have $x_j^{(i)} = x_j^{(i+1)}$ <u>for all j but one</u>, i.e., there is <u>one and only one</u> $j_0 \in \{1, \ldots, k\}$ such that $x_{j_0}^{(i)} \neq x_{j_0}^{(i+1)}$.

Differential inclusions as a small-step limit of substitution sequences:

Let $\underline{x}^{(1)}, \underline{x}^{(2)}, \ldots$ be a k-morphic substitution sequence.
Suppose that $\delta > 0$ is the maximum mutation step, i.e., $|x_j^{(i)} - x_j^{(i+1)}| \leq \delta \quad \forall\, i, j$.

Then, for any $N, i \geq 1$ and $j \in \{1, \ldots, k\}$ we have

$$0 \leq S_{\underline{x}^{(N+i)}}(x_j^{(N+i+1)}) =$$

(write $S'_{\underline{x}^{(N)}}(x_j) := \partial_y S_{\underline{x}^{(N)}}(y)\big|_{y=x_j}$)

$$= S'_{\underline{x}^{(N+i)}}(x_j^{(N+i)})(x_j^{(N+i+1)} - x_j^{(N+i)}) + O(\delta^2) =$$

$$= \left(S'_{\underline{x}^{(N)}}(x_j^{(N)}) + i\,O(\delta)\right)(x_j^{(N+i+1)} - x_j^{(N+i)}) + O(\delta^2) =$$

$$= S'_{\underline{x}^{(N)}}(x_j^{(N)})\left(x_j^{(N+i+1)} - x_j^{(N+i)}\right) + (i+1)\,O(\delta^2)$$

Hence,

$$0 \leq \sum_{i=0}^{n-1} \left[S'_{\underline{x}^{(N)}}(x_j^{(N)})\left(x_j^{(N+i+1)} - x_j^{(N+i)}\right) + (i+1)\,O(\delta^2) \right] =$$

$$= S_{\underline{x}^{(N)}}(x_j^{(N)})(x_j^{(N+n)} - x_j^{(N)}) + \tfrac{1}{2} n(n+1)\,O(\delta^2)$$

Define the interpolating function $\underline{x}_\delta : \mathbb{R}_+ \to \mathbb{R}^k$ such that $\underline{x}_\delta(i\delta) = \underline{x}^{(i)}$ $\forall i$.

Then

$$0 \leq S'_{\underline{x}_\delta(N\delta)}(x_{\delta,j}(N\delta))\left(x_{\delta,j}(N\delta+n\delta) - x_{\delta,j}(N\delta)\right) +$$

$$+ \tfrac{1}{2} n(n+1)\, O(\delta^2)$$

and hence

$$0 \leq S'_{\underline{x}_\delta(N\delta)}(x_{\delta,j}(N\delta)) \, \frac{x_{\delta,j}(N\delta+n\delta) - x_{\delta,j}(N\delta)}{n\delta} + \tfrac{1}{2}(n+1)\, O(\delta)$$

Letting $N, n \to \infty$ and $\delta \downarrow 0$ such that $n\delta \to 0$, $N\delta \to t > 0$ and $\underline{x}_\delta \to \underline{x}$ (the latter only point-wise), then we get the <u>differential inclusion</u>

$$\boxed{0 \leq S'_{x(t)}(x_j(t))\, \dot{x}_j(t) \quad \forall j}$$

(d)

(where $S'_x(x) := \partial_y S_x(y)\big|_{y=x}$).

Solution concept

$\underline{x} : \mathbb{R}_+ \longrightarrow \mathbb{R}^k$ is a solution of the differential inclusion

(2) $$\boxed{0 \leq S'_{\underline{x}}(x_i)\, \dot{x}_i \quad \forall i}$$

if $\underline{x}$ is continuous and piece-wise differentiable and satisfies (2) wherever it is differentiable.

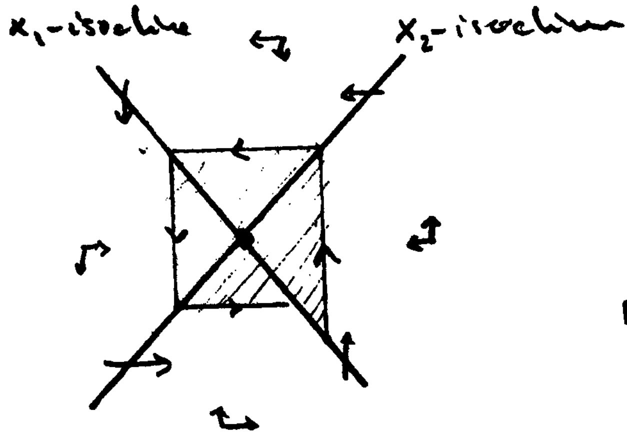
Stability

$\underline{x}^*$ is *stable* if for every nbd. U of $\underline{x}^*$ there exists a nbd V of $\underline{x}^*$ such that *every* solution of (2) starting in V will stay in U for all $t > 0$.

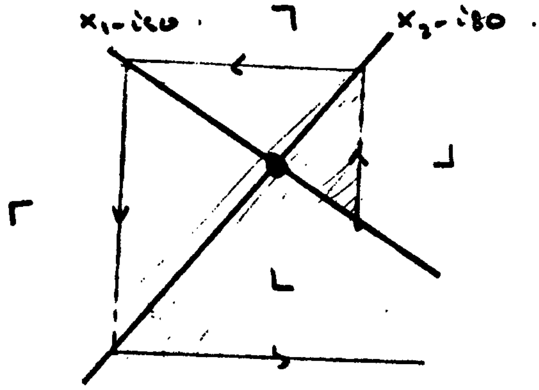
(i.e., solutions starting close to $\underline{x}^*$ will stay close to $\underline{x}^*$ for all $t>0$.)

In the context of adaptive dynamics, we call this type of stability "*total stability*".

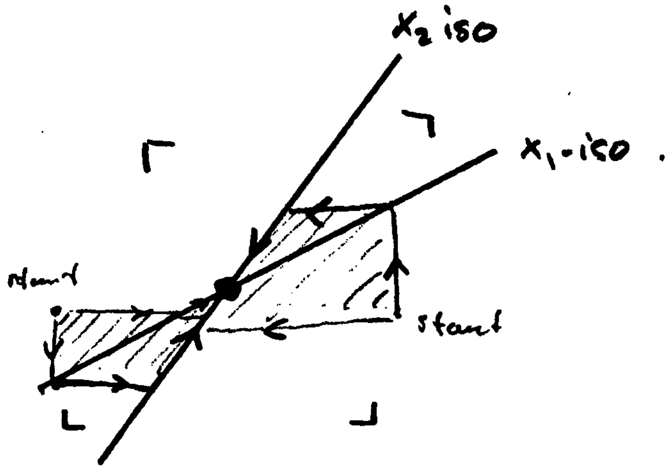
Graphical examples.



total stability.



not totally stable.



total stability.

Also see:
Matessi &
Di Pasquale
J.M.B (1996)

Linearized diff. incl.:

Linearize $S'_{\underline{x}}(x_j)$ near a singular point $\underline{x}^*$:

③ $S'_{\underline{x}}(x_i) \doteq \sum_{j=1}^{k} a_{ij}(x_j - x_j^*)$

where

④ $a_{ij} := \left[\partial_{yy} S_{\underline{x}}(y)\right]_{\substack{y=x_i^* \\ \underline{x}=\underline{x}^*}} \delta_{ij} + \left[\partial_{x_j y} S_{\underline{x}}(y)\right]_{\substack{y=x_i^* \\ \underline{x}=\underline{x}^*}}$

(Kronecker-δ: $\delta_{ij} = 1 \Leftrightarrow i=j$)

Substitution of ③ into ② gives the linear differential inclusion

⑤ $$\boxed{0 \leqslant (x_i - x_i^*)^{\cdot} \sum_{j=1}^{k} a_{ij}(x_j - x_j^*) \quad \forall i}$$

We study the stability of $\underline{x}^*$ in the linear diff. incl. ⑤:

Theorem.

Consider the lin. diff. incl. (5)

(i) If there exist $d_1, \ldots, d_k > 0$ such that

$$\boxed{\frac{a_{ii}}{d_i} < - \sum_{j \neq i} \frac{|a_{ij}|}{d_j} \quad \forall i}$$

(i.e., the matrix (a_{ij}) is strictly negative diagonally dominant)

then

$$\boxed{\left(\max_{\forall i} d_i |x_i - x_i^*| \right)^{\bullet} \leq 0}$$

and hence $\underline{x}^*$ is stable.

(ii) If there exist $d_1, \ldots, d_k > 0$ such that

$$\boxed{\frac{a_{ii}}{d_i} > \sum_{j \neq i} \frac{|a_{ij}|}{d_j} \quad \forall i}$$

(i.e., the matrix (a_{ij}) is strictly positive diagonally dominant.)

then

$$\boxed{\left(\max_{\forall i} d_i |x_i - x_i^*| \right)^{\bullet} \geq 0}$$

and hence $\underline{x}^*$ is repelling.

Proof

Take $i \in \{1, \dots, k\}$ such that

$$\boxed{d_i |x_i - x_i^*| = \max_{\forall j} d_j |x_j - x_j^*| > 0}.$$

Then:

$$0 \leq (x_i - x_i^*) \sum_{j=1}^{k} a_{ij}(x_j - x_j^*)$$

$$\Longleftrightarrow \quad 0 \leq (x_i - x_i^*)^{\bullet} (x_i - x_i^*) \left(a_{ii} + \sum_{j \neq i} a_{ij} \frac{x_j - x_j^*}{x_i - x_i^*} \right)$$

$$\Longleftrightarrow \circledast \quad \boxed{0 \leq \left[(x_i - x_i^*)^2 \right]^{\bullet} \left(\frac{a_{ii}}{d_i} + \sum_{j \neq i} \frac{a_{ij}}{d_j} \frac{d_j (x_j - x_j^*)}{d_i (x_i - x_i^*)} \right)}$$

To prove part (i) of the Theorem, assume that (a_{ij}) is strictly negative diagonally dominant.

Then:

$$\frac{a_{ii}}{d_i} < - \sum_{j \neq i} \frac{|a_{ij}|}{d_j} \leq - \sum_{j \neq i} \frac{|a_{ij}|}{d_j} \frac{d_j |x_j - x_j^*|}{d_i |x_i - x_i^*|}$$

$$\leq - \sum_{j \neq i} \frac{a_{ij}}{d_j} \frac{d_j |x_j - x_j^*|}{d_i |x_i - x_i^*|}$$

$$\Longrightarrow \quad \boxed{\frac{a_{ii}}{d_i} + \sum_{j \neq i} \frac{a_{ij}}{d_j} \cdot \frac{d_j (x_j - x_j^*)}{d_i (x_i - x_i^*)} < 0}$$

But then, from ⊛, we have

$$\left[(x_i - x_i^*)^2\right]^{\bullet} \leq 0$$

$$\Leftrightarrow \quad \left(d_i |x_i - x_i^*| \right)^{\bullet} \leq 0$$

which, by choice of i (see top of previous page) is equivalent to

$$\boxed{\left(\max_{\forall j} d_j |x_j - x_j^*| \right)^{\bullet} \leq 0}$$

which is what we set out to prove.

The proof of part (ii) of the Theorem is very similar and is left as an exercise.

∎

Special case $k=1$

If $k=1$, we have

$$a_{11} = \partial_{yy} S_x(y)\Big|_{y=x=x^*} + \partial_{xy} S_x(y)\Big|_{y=x=x^*} =$$

$$= \frac{1}{2}\left(\partial_{yy} S_x(y)\Big|_{y=x=x^*} - \partial_{xx} S_x(y)\Big|_{y=x=x^*}\right)$$

(see notes on the classification of the loc. config's of the PIP)

$$= \frac{1}{2}\left(C_{22} - C_{11}\right)$$

Consequently, we have:

x^* is stable if $C_{22} < C_{11}$, and
x^* is repelling if $C_{22} > C_{11}$

Which is exactly the same as what we found in the classification of the local configuration of the PIP.

Special case $k=2$.

Suppose $k=2$.

(i) Then strict neg. diag. dom. means

$$\exists\, d_1, d_2 > 0 : \frac{a_{11}}{d_1} < -\frac{|a_{12}|}{d_2} \quad \& \quad \frac{a_{22}}{d_2} < -\frac{|a_{21}|}{d_1}$$

take $\delta := \frac{d_1}{d_2}$

$$\Longleftrightarrow \exists\, \delta > 0 : a_{11} < -\delta\,|a_{12}| \quad \& \quad a_{22} < \frac{-1}{\delta}|a_{21}| \leq 0$$

$$\Longleftrightarrow \exists\, \delta > 0 : \frac{-a_{11}}{|a_{12}|} > \delta > 0 \quad \& \quad \delta > -\frac{|a_{21}|}{a_{22}} \geq 0$$

$$\Longleftrightarrow \exists\, \delta > 0 : 0 \leq -\frac{|a_{21}|}{a_{22}} < \delta < -\frac{a_{11}}{|a_{12}|}$$

$$\Longleftrightarrow \boxed{\begin{array}{l} a_{11} < 0 \\ a_{22} < 0 \\ a_{11}a_{22} > |a_{12} \cdot a_{21}| \end{array}} \overset{\text{Th.}}{\Longrightarrow} \underline{x^* \text{ is stable.}}$$

(ii) Likewise, strict neg. diag. dom. means:

$$\boxed{\begin{array}{l} a_{11} > 0 \\ a_{22} > 0 \\ a_{11}a_{22} > |a_{12}a_{21}| \end{array}} \overset{\text{Th.}}{\Longrightarrow} \underline{x^* \text{ is repelling}}$$