Appendix B

Theory of Poincaré and Bendixon

$$(*)\begin{cases} \dfrac{dx}{dt} = f(x,y) \\[2ex] \dfrac{dy}{dt} = g(x,y) \end{cases}$$

system in $\mathbb{R}^2$, f, g cont. diff.

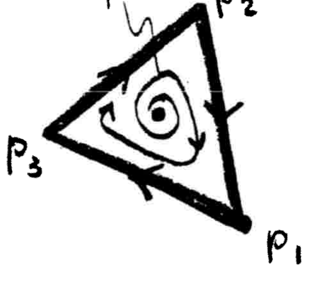
Some definitions.

- Positive semi-orbit (forward orbit) through point (x_0, y_0) is the set $\Gamma_+ = \{(x(t), y(t)) : t \geq 0 \text{ and } (x(0), y(0)) = (x_0, y_0)\}$
- Negative semi-orbit (backward orbit) through (x_0, y_0) is the set $\Gamma_- = \{(x(t), y(t)) : t \leq 0 \text{ and } (x(0), y(0)) = (x_0, y_0)\}$
- (Full) orbit $\Gamma = \Gamma_+ \cup \Gamma_-$.
- $p \in \omega$-limit of Γ iff $\exists\, t_1 < t_2 < \ldots \to \infty$ such that ~~which~~ $(x(t_n), y(t_n)) \to p$ as $n \to \infty$.
- $p \in \alpha$-limit of Γ iff $\exists\, t_1 > t_2 > t_3 \to -\infty$ such that $(x(t_n), y(t_n)) \to p$ as $n \to \infty$.

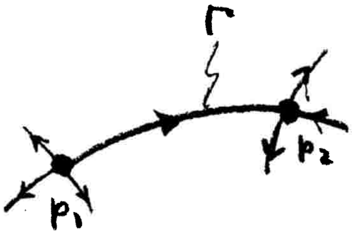
ω-limit $(\omega(x_0, y_0))$ = the set of limit points of the function $t \longmapsto (x(t), y(t))$

α-limit same for $t \longmapsto (x(-t), y(-t))$.

Examples



- ω-limit of Γ is the heteroclinic cycle consisting of p_1, p_2, p_3 and connecting orbits.
- α-limit of Γ is the unstable focus in the middle.



ω-limit is $\{p_2\}$,
α-limit is $\{p_1\}$.

Here the ω-limit of Γ is a closed orbit.

Theorem. (Poincaré-Bendixon)

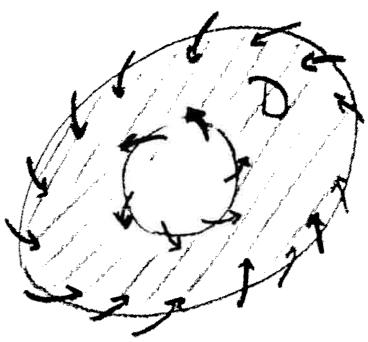
The ω-limit of an orbit Γ in $\mathbb{R}^2$ of which Γ_+ is bounded is either a periodic orbit or contains an equilibrium.

$\boxtimes$

Corollary

Suppose D is a closed and bounded and forward-invariant region in $\mathbb{R}^2$ that contains no equilibrium. Then D contains a periodic orbit.

$\boxtimes$



Thus, D must contain a periodic orbit if it does not contain an equilibrium.

B3

Line integral of a vector field.

$F: \mathbb{R}^2 \to \mathbb{R}^2$ cont. diff. vector field $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$

$\gamma: [a,b] \to \mathbb{R}^2$ cont differentiable curve.

$$\int_\gamma F = \boxed{\int_\gamma F.ds := \int_a^b F(\gamma(t)) \cdot D\gamma(t)\, dt}$$

where

$$F \cdot D\gamma = F_1 D\gamma_1 + F_2 D\gamma_2 \quad \text{(scalar product).}$$

Green's Theorem.

Let $F(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$ be a cont. differentiable vectorfield on a closed & bounded and simply connected region (no holes) $D \subset \mathbb{R}^2$ with smooth (cont. diff) boundary ∂D. Then:

$$\int_{\partial D} F.ds = \cancel{\int_{\partial D} (f_1 dx + f_2 dy) = \iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx\,dy}$$

(notation) $\int_{\partial D} (f_1 dx + f_2 dy) = \iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx\,dy$

(real valued function)

$\blacksquare$

Definition.

The divergence of a vector field $F(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$ is $\operatorname{div} F := \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}$ (i.e., the trace of the Jacobi-matrix)

$\blacksquare$

By

Bendixon's theorem.

Let $B \subset \mathbb{R}^2$ be a simply connected region and let $F(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$ be a continuously differentiable vector field on B.

If $\operatorname{div} F \geq 0$ on B or $\operatorname{div} F \leq 0$ on B, but $\operatorname{div} F$ is not identical zero on B, then the ODE

$$(*) \begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{cases}$$

has no periodic orbit in B

Proof. (by contradiction) $\sigma : [0,T] \to \Gamma$.

Suppose B contains a periodic orbit Γ with period T, and let D be the region enclosed by Γ.

$$\iint_D \operatorname{div} F \, dx\,dy = \iint_D \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dx\,dy \overset{\text{Green}}{=}$$

$\int_\Gamma \begin{pmatrix} -g \\ f \end{pmatrix} \cdot ds$

$$= -\int_\Gamma (g\,dx - f\,dy) = -\int_0^T \left(g \frac{dx}{dt} - f \frac{dy}{dt} \right) dt =$$

$$= -\int_0^T \left(\frac{dy}{dt} \frac{dx}{dt} - \frac{dx}{dt} \frac{dy}{dt} \right) dt = 0.$$

But this contradicts the assumptions on the sign of $\operatorname{div} F$ on B and hence on D.

Corollary (Dulac)

Situation as in Bendixon's theorem, but now assume that there exists a positive continuously differentiable function $u: B \to (0,\infty)$ such that $\operatorname{div} uF \geq 0$ on B or $\operatorname{div} uF \leq 0$ on B, but such that $\operatorname{div} uF$ is not identical zero on B.

Then B does not contain a periodic orbit.

Proof

(Suppose $\exists$ limit cycle Γ. Then $\Gamma \cup$ interior is forward invariant $\Rightarrow \tau$ (below) is well defined $\forall t \geq 0$)

Non-linear scaling of time:

$$\tau := \int_0^t \frac{ds}{u(x(s), y(s))}$$

Then

$$(**) \quad \begin{cases} \dfrac{dx}{d\tau} = u(x,y)\, f(x,y) \\[2ex] \dfrac{dy}{d\tau} = u(x,y)\, g(x,y) \end{cases}$$

> The function u is called the *Dulac-function*

Claim follows from Bendixon's theorem applied to (**). ∎